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# Information in the Term Structure of Yield Curve Volatility

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**ABSTRACT**

Using a novel no-arbitrage model and extensive second-moment data, we decompose conditional volatility of U.S. Treasury yields into volatilities of short-rate expectations and term premia. Short-rate expectations become more volatile than premia before recessions and during asset market distress. Correlation between shocks to premia and shocks to short-rate expectations is close to zero on average and varies with the monetary policy stance. While Treasuries are nearly unexposed to variance shocks, investors pay a premium for hedging variance risk with derivatives. We illustrate the dynamics of the yield volatility components during and after the financial crisis.

IN THIS PAPER, WE CHARACTERIZE THE DYNAMICS of the conditional second moments in the U.S. Treasury market in terms of volatilities of short-rate expectations and term premia, as well as their covariance. A vast literature decomposes the information contained in nominal Treasury yields to study term premia and short-rate expectations. With a few important exceptions,[1] much of the recent work has approached this question assuming constant volatility, largely due to tractability and the known difficulties with the joint modeling of the first and second moments of yields. Consequently, less is known about the economic properties of Treasury market volatility. Intuition suggests, however, that volatility can reveal new information that is hard to extract from yield levels alone. For example, to the extent that expected short rates reflect market expectations about the path of monetary policy, their volatility can reveal the amount of uncertainty surrounding that path. Further, while it is a priori unclear how term premia and expected short rates covary over time, studying

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[1] See, for example, Almeida, Graveline, and Joslin (2011), Collin-Dufresne and Goldstein (2002), Collin-Dufresne, Goldstein, and Jones (2009), Joslin (2010, 2014), and Kim and Singleton (2012). We review this literature in more detail below.

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their joint conditional dynamics can cast light on the dependence between shocks influencing short- and long-term yields. These and related questions underscore the importance of understanding the sources of volatility in the yield curve.

Relative to previous literature, our approach introduces two new elements. First, to identify the empirical properties of interest rate volatility, we rely on nearly 20 years' worth of high-frequency data from the U.S. Treasury market, which we use to construct the realized covariance matrix of zero-coupon yields across maturities. This step is important because, by making volatility essentially observable, we mitigate concerns about its misspecification. We augment our analysis with information from the options market, which reflects risk-adjusted expectations of volatility at several points along the yield curve. Second, we propose a no-arbitrage term structure model that accommodates multivariate dynamics of yield volatilities found in the data. As a new feature of our approach, we model the entire stochastic covariance matrix of risks in the yield curve without sacrificing the flexibility that a Gaussian model offers for studying the term premia and short-rate expectations. To reliably identify the parameters underlying our volatility decomposition, we estimate the model with interest rate surveys, yields, and their realized and implied volatilities. We argue that informative second-moment data and a flexible model complement each other, and both are necessary elements of our analysis.

Interest rate volatility moves around due to changes in volatilities of short-rate expectations, term premia, and their covariance. Using our model, we are able to analyze the importance of each of these terms. We show that, while volatility of short-term yields is mostly determined by the component due to short-rate expectations, the contribution of term premium volatility increases with maturity. Volatile short-rate expectations rather than term premia produce the well-known hump in yield volatilities at maturity of two to three years. This is consistent with the view shared by practitioners and policy makers that this segment of the yield curve is sensitive to market expectations about the future short rate and, by extension, about the path of monetary policy. In terms of time-series properties, we find that the short-rate expectations component of volatility is shorter lived and less persistent, with a half-life below 10 weeks, which is less than half that of the term premium volatility.

Short-rate expectations become increasingly volatile in anticipation of recessions and during periods of distress in asset markets, which in our sample coincide with monetary policy easing. The reaction of term premium volatility is relatively more muted and has a less clear business cycle pattern. A case in point is the financial crisis of 2007/09. Volatility of short-rate expectations rises dramatically in the early summer of 2007 when symptoms of the subprime crisis appear and starts to decline as the Fed enters the zero lower bound in December 2008. At the same time, term premium volatility begins to rise only in the fall of 2008 after the Lehman collapse, and remains elevated though the end of our sample in December 2010. Its peak in March 2009 coincides with the Fed announcing an extension of the quantitative easing (QE) to include long-term Treasuries. These results indicate that, in the aftermath of the financial crisis, the Fed was able to reduce uncertainty about the path of the short rate, and the QE policies have led to an increase in the volatility of the term premia.

By modeling the stochastic covariance of risks in the yield curve, we are able to study the comovement between shocks to short-rate expectations and shocks to term premia over time. The model implies that, over the last two decades, the conditional correlation between those shocks is on average close to zero and varies over time in relation to the stance of the monetary policy. For example, the correlation increases during the long easing episode in 2000/03 and declines afterwards. The model interprets Greenspan's conundrum period (i.e., the apparent lack of reaction of long-term interest rates to the tightening of monetary policy in 2004/06) as a gradual decline in the correlation between shocks to short-rate expectations and shocks to term premia.

As part of our analysis, we study how investors price risks in Treasury bonds and in the fixed income market more broadly. The model distinguishes between shocks to the levels and shocks to the variances of yields, and thus allows us to analyze both the term premia and the variance risk premia. We find that Treasury yields are nearly unexposed to the variance shocks and thus investors in Treasury bonds are compensated almost exclusively for facing the yield curve shocks. While this finding is consistent with earlier literature (summarized below) documenting that Treasury yields do not span volatility risk, it does not imply that volatility risk is not priced in the fixed income market. Indeed, the model suggests that investors are willing to pay a large premium for hedging variance risk via interest rate derivatives, with the average conditional Sharpe ratio reaching up to $-1.7$ per annum. We decompose the variance risk premium into compensation for facing three types of shocks: shocks to the variance of short-rate expectations, shocks to the variance of term premia, and shocks to the corresponding covariance. The decomposition attributes the main source of the premium to shocks associated with the variance of short-rate expectations and shocks to the covariance between short-rate expectations and term premia. To the extent that volatility of short-rate expectations reflects uncertainty about the path of monetary policy, our results suggest that hedging such uncertainty is an important consideration for fixed income investors.

Much of the term structure literature studies low-dimensional affine term structure models (ATSMs) with a single factor driving the second moments of yields, such as $A_1(3)$ or $A_1(4)$ models in the classification of Dai and Singleton (2002).[2] When estimated using only yield data, volatility dynamics implied by those models correspond poorly to model-free volatility measures. The misspecification becomes especially visible at long maturities and in the post-Volcker period, where model-implied volatilities have been found to be either uncorrelated or negatively correlated with realized volatility or GARCH-type estimates (Collin-Dufresne, Goldstein, and Jones (2009), Jacobs and Karoui (2009)). Based on several different models with a single volatility factor estimated on Japanese yields, Kim and Singleton (2012) report similar difficulties

[2] Notation $A_m(n)$ means an affine model with $n$ factors in total, of which $m$ are square-root (positive) processes and $n - m$ are conditionally Gaussian.

in fitting volatility at long maturities and conclude that more than one volatility factor may be necessary to match the data. Using interest rate caps in a 1995 to 2006 sample, Almeida, Graveline, and Joslin (2011) show that derivatives help identify volatility states in an affine setting. However, given their focus on three-factor models, the relative errors they report for fitting caps are large.

There are few established results about the economic sources of interest rate volatility in the literature. Collin-Dufresne and Goldstein (2002, CDG) argue that volatility risk cannot be hedged by Treasury bonds and thus volatility states are difficult to extract from the term structure of interest rates alone—a phenomenon they call "unspanned stochastic volatility" (USV). Andersen and Benzoni (2010) confirm this finding using realized volatility estimated from high-frequency Treasury yields.[3] To break the spanning of volatility risk that arises in standard ATSMs, CDG propose a set of parametric USV restrictions. However, Joslin (2014) shows that those restrictions constrain other aspects of yield dynamics in a way that is empirically undesirable and is rejected by the data.

We contribute to this discussion by proposing a term structure model that, consistent with the data, accommodates the multivariate properties of yield volatility. Rather than imposing USV-type restrictions, we estimate the model using information obtained from the realized and implied second moments of yields. We find that together these two elements—a flexible model and our data—alleviate the problems documented in previous studies.

A related literature introduces stochastic volatility into macro-finance term structure models, but understanding the sources of yield volatility per se is not its direct focus (e.g., Haubrich, Pennacchi, and Ritchken (2012), Campbell, Sunderam, and Viceira (2013)). Here, our objective is to characterize the yield volatility that arises due to the variation in term premia, short-rate expectations, and their conditional covariance, as well as to study how investors price shocks to each of those components.

The remainder of this paper is organized as follows. Section I discusses our data and the construction of the realized and implied yield volatility measures. Section II presents the model, and Section III verifies its empirical performance in terms of time-series and cross-sectional fit to volatilities. Section IV decomposes yield volatilities across maturities into components induced by short-rate expectations and term premia. Section V studies the market pricing of shocks to the levels and variances of yields, respectively. Section VI contains additional robustness results. Section VII concludes.

[3] Several other papers document a weak relation between the bond volatility, realized as well as derivative-based, and the spot yield curve factors. See, for example, Heidari and Wu (2003) and Li and Zhao (2006).

## I. Data

### A. High-Frequency Bond Data and Zero-Coupon Yield Curve Tick-by-Tick

We obtain 19 years' worth of high-frequency data on the pricing of U.S. Treasury securities from January 1992 through December 2010 by splicing historical observations from two inter-dealer broker platforms: GovPX (1992:01 to 2000:12) and BrokerTec (2001:01 to 2010:12).[4] The merged data set covers about 60% of the transactions in the secondary U.S. Treasury market. GovPX comprises Treasury bills and bonds with maturities of three, six, and 12 months, as well as two, three, five, seven, 10, and 30 years. BrokerTec contains only Treasury bonds with maturities of two, three, five, seven (in part of the sample), 10, and 30 years. In the GovPX period, we identify on-the-run securities and use their mid-quotes for further analysis. Unlike GovPX, which is a voice-assisted brokerage system, BrokerTec is a fully electronic trading platform attracting vast liquidity and thus allowing us to consider traded prices of on-the-run securities. Roughly 95% of trading occurs between 7:30AM and 5:00PM EST (e.g., Fleming (1997)), which we treat as the trading day. The raw data set contains coupon bonds. We sample bond prices at 10-minute intervals taking the last available price within the interval, and convert them into zero-coupon yields following the procedure of Fisher, Nychka, and Zervos (1994). More information about our data and technical details on the construction of zero-coupon yields are available in the Internet Appendix.[5]

### B. Realized Yield Covariance Matrix

Observing the zero-coupon yield curve at a high frequency allows us to estimate the realized covariance matrix of yields. Let $y_t$ be the vector of zero-coupon yields with different maturities observed at time $t$. Time is measured in daily units. The realized covariance matrix is constructed by summing the outer products of a vector of 10-minute yield changes, and aggregating them over the interval of one day $[t, t+1]$:

$$RCov(t, t+1; N) = \sum_{i=1,\ldots,N} \left(y_{t+\frac{i}{N}} - y_{t+\frac{i-1}{N}}\right)\left(y_{t+\frac{i}{N}} - y_{t+\frac{i-1}{N}}\right)', \qquad (1)$$

where $N = 58$ is the number of equally spaced yield observations per day $t$ at the 10-minute sampling, and $i$ denotes the $i^{\text{th}}$ change during the day. For frequent sampling, the quantity (1) converges to the quadratic co-variation of yields (Jacod (1994), Barndorff-Nielsen and Shephard (2004)). Weekly or monthly realized covariances follow by aggregating the daily measure over the corresponding time interval. To obtain annualized numbers, we multiply $RCov$

[4] GovPX started operating in 1991. See Mizrach and Neely (2006) and Fleming and Mizrach (2009) for a description of the BrokerTec and GovPX platforms.

[5] The Internet Appendix is available in the online version of the article on the *Journal of Finance* website.

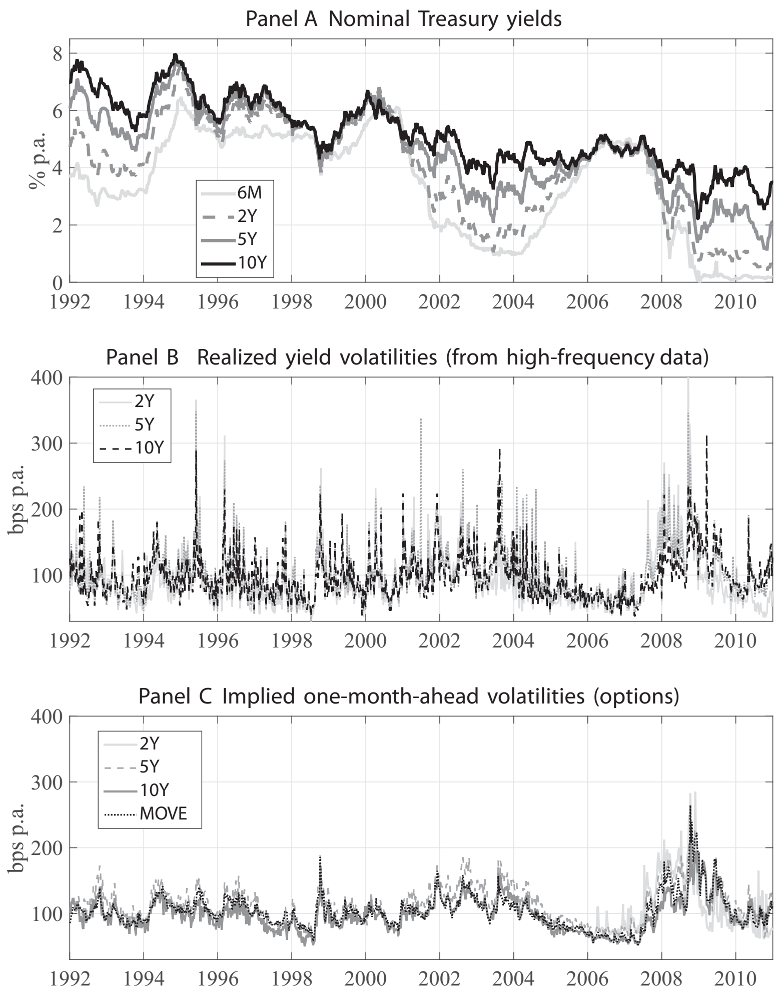


**Figure 1. Time series of yields and volatilities.** The figure plots the time series of yields (Panel A), realized volatilities (Panel B), and implied volatilities (Panel C). We superimpose implied volatilities with the MOVE index. Yields are reported in percentage per annum and volatilities in basis points per annum. The data are sampled weekly. The sample covers the 1992:01 to 2010:12 period, except for the two-year implied volatility, which starts in March 2006.

by 250 for the daily, 52 for the weekly, and 12 for the monthly frequency.[6] Realized volatility is the square root of the realized variance. In constructing (1), we consider yields with maturities of two, three, five, seven, and 10 years, which attract most of the liquidity in the secondary bond market (e.g., Fleming and Mizrach (2008)). Figure 1 plots zero-coupon yields (Panel A) and

[6] In the online Appendix, we verify the robustness of this estimator, and compare it to alternatives proposed in the literature (e.g., Hayashi and Yoshida (2005)).

realized volatilities (Panel B) for maturities of two, five and 10 years at a weekly frequency.

To ensure that our volatility measures reflect views of active participants in the Treasury market rather than institutional effects, we limit construction of $RCov$ to within-day observations, excluding the volatility outside the U.S. trading hours.[7] Further, we focus on maturities of two years and above. Over our sample period, the very short end of the yield curve (T-bills) exhibited a gradual decline in trading activity, with high-frequency data available only until March 2001.[8] Moreover, dynamics of the short-end of the curve have been shown to be confounded by money market noise (Piazzesi (2005)), T-bill-specific factors not shared by longer-maturity Treasury securities (Duffee (1996)), and institutional effects (Hilton (2005)), distortions not directly relevant to the analysis we perform.

### C. Implied Interest Rate Volatilities

We obtain implied volatilities from the end-of-day prices of individual options on two-, five- and 10-year Treasury bond and note futures and the corresponding underlyings from the Chicago Mercantile Exchange (CME). For maturities of five and 10 years, we are able to construct implied volatilities covering the entire period from 1992 through 2010. For the two-year maturity, we can reliably construct implied volatilities beginning only in March 2006. The underlying of the option is futures on a hypothetical Treasury bond or note that pays coupons. While we label the implied volatility using the maturity of the underlying bond (e.g., two-year implied volatility refers to the volatility from options written on the two-year bond futures), for model estimation in Section II.C we convert it to a zero-coupon equivalent. We use options that are closest to at-the-money and to the one-month maturity, as those represent the most active part of the market and provide an accurate approximation to the risk-neutral expectation of the yield volatility during the following month (e.g., Carr and Wu (2006)). Implied volatilities are obtained using the Black model. Similar to realized yield volatilities, we report annualized implied volatilities on a yield basis. This convention is also followed by the bond market volatility benchmark index, the Merrill Lynch Option Volatility Estimate (MOVE), which averages

[7] We observe several spikes in the between-day volatility, which we cannot relate to any major news in the U.S. market. To account for the total magnitude of volatility, we add to the within-day number the squared overnight yield change from close (5:00PM) to open (7:30AM). We then compute the unconditional average of the total and within-day $RCov$, and each day scale the within-day $RCov$ dynamics by the total-to-within ratio. This procedure follows Andersen and Benzoni (2010).

[8] This is when the GovPX sample ends and we switch to the BrokerTec database. BrokerTec does not contain maturities below two years. Alternatively, to cover the short-maturity segment one could consider high-frequency T-bill futures. However, the number of trades in the T-bill futures contract at the Chicago Mercantile Exchange has consistently declined over our sample period, reaching as few as five trades per day on average in 2003 (see the Internet Appendix for details). For this reason, standard sources of high-frequency data (e.g., Tickdata.com) have stopped supplying T-bill futures after 2003, quoting low secondary market activity.

implied volatilities across several bond maturities.[9] Panel C of Figure 1 plots the implied volatility series and superimposes them with the MOVE index. Details about options data, conversions, and robustness of the implied volatility series are contained in the Internet Appendix.

### *D. Descriptive Statistics*

Table I reports summary statistics for zero-coupon yields (Panel A), realized and implied volatilities (Panel B), and unconditional correlations between yields and volatilities (Panel C). The data are sampled at the last business day of a week. Relative to the implied volatilities, realized volatilities are less persistent and have a higher unconditional standard deviation. This is consistent with implied volatilities reflecting risk-adjusted conditional market expectations.[10]

Panel C of Table I points to a slightly negative correlation between the level of yields and volatilities in our sample. This is different from the Cox, Ingersoll, and Ross (CIR, 1985) model, which predicts that volatility is high whenever the short-term rate is high. While this prediction had empirical support in the high-inflation period of the late 1970s and early 1980s (Chan et al. (1992)), recent USV literature argues that the relationship between yields and volatilities is weak. Our estimates are broadly consistent with this conclusion.

The bottom section of Panel C reports unconditional correlations between realized and implied volatilities and the MOVE index. The lowest correlation is 0.51 between the 10-year implied and the two-year realized volatility. The correlation between MOVE and our implied volatility series exceeds 0.9, except for the two-year maturity where we cover only the recent part of the sample due to data limitations.

Finally, we find that, similar to yield levels, variation in yield volatilities can be described by three factors. The first three principal components explain 90%, 7%, and 1.5% of the realized volatilities with maturities between two and 10-years, and 79%, 13%, and 5% of realized volatilities jointly with the five- and 10-year implied volatilities. At the weekly frequency, the first three principal components of realized volatilities explain about 50% of the variation in implied volatilities on average across maturities. This share increases to about 70% when we smooth realized volatilities with a four-week moving average.

[9] MOVE is a weighted average of implied volatilities of one-month options on Treasury bonds. The weights are 20%, 20%, 40%, and 20% for the two-, five-, 10-, and 30-year maturities, respectively. Options used by MOVE are written on cash bonds rather than the Treasury futures that we use. They are traded on an OTC basis, and are therefore less liquid. The data are available only from proprietary sources such as large broker dealers.

[10] Since implied volatilities have coupon bonds as the underlying, their mean levels are not directly comparable to the realized volatilities that pertain to zero-coupon yields. While this fact complicates inference about the variance risk premium using raw data, we are able to study the properties of this premium based on the model we introduce below.

**Table I**
## Descriptive Statistics

The table presents descriptive statistics for yields (Panel A) and realized yield volatilities (Panel B). Implied volatility statistics in Panel B are for coupon bonds, which is marked with the superscript "($c$)" in the column heading. Panel C reports the unconditional correlations between yields, realized volatilities, and implied volatilities, as well as with the Merrill Option Implied Volatility (MOVE) index. The sample period is 1992:01 to 2010:12, except for the implied volatility on a two-year bond (marked with $\star$), for which the data begin in March 2006. The frequency of the data is weekly. AR(1m) denotes the autoregressive coefficient estimated at the monthly horizon with overlapping weekly data.

Panel A. Yields (% p.a.)

| | 6M | 2Y | 3Y | 5Y | 7Y | 10Y |
|---|---|---|---|---|---|---|
| Mean | 3.39 | 3.97 | 4.23 | 4.63 | 4.90 | 5.18 |
| Skew | −0.39 | −0.37 | −0.36 | −0.25 | −0.09 | 0.13 |
| Kurt | 1.84 | 2.00 | 2.11 | 2.20 | 2.24 | 2.28 |
| AR(1m) | 0.998 | 0.995 | 0.992 | 0.985 | 0.980 | 0.975 |

Panel B. Volatilities (bps p.a.)

| Realized Volatilities | 2Y | 3Y | 5Y | 7Y | 10Y |
|---|---|---|---|---|---|
| Mean | 93.19 | 99.98 | 101.61 | 97.81 | 96.44 |
| Stdev | 43.63 | 46.29 | 41.41 | 36.95 | 34.59 |
| Skew | 2.08 | 2.07 | 1.65 | 1.35 | 1.53 |
| Kurt | 10.89 | 10.45 | 7.61 | 5.80 | 7.31 |
| AR(1m) | 0.44 | 0.39 | 0.41 | 0.44 | 0.42 |
| Implied Volatilities | 2Y$^{\star(c)}$ | | 5Y$^{(c)}$ | | 10Y$^{(c)}$ |
| Mean | 112.71 | | 115.39 | | 98.72 |
| Stdev | 45.04 | | 29.43 | | 24.55 |
| Skew | 1.13 | | 0.64 | | 0.69 |
| Kurt | 3.79 | | 3.58 | | 4.05 |
| AR(1m) | 0.71 | | 0.81 | | 0.83 |

(*Continued*)

**Table I**—*Continued*

Panel C. Unconditional Correlations of Yields, Realized Volatilities, and Implied Volatilities

| | $y_t^{6M}$ | $y_t^{2Y}$ | $y_t^{3Y}$ | $y_t^{5Y}$ | $y_t^{7Y}$ | $y_t^{10Y}$ | $RVol_t^{2Y}$ | $RVol_t^{3Y}$ | $RVol_t^{5Y}$ | $RVol_t^{7Y}$ | $RVol_t^{10Y}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y_t^{6M}$ | 1.00 | . | . | . | . | . | . | . | . | . | . |
| $y_t^{2Y}$ | 0.97 | 1.00 | . | . | . | . | . | . | . | . | . |
| $y_t^{3Y}$ | 0.95 | 0.99 | 1.00 | . | . | . | . | . | . | . | . |
| $y_t^{5Y}$ | 0.88 | 0.96 | 0.98 | 1.00 | . | . | . | . | . | . | . |
| $y_t^{7Y}$ | 0.81 | 0.91 | 0.95 | 0.99 | 1.00 | . | . | . | . | . | . |
| $y_t^{10Y}$ | 0.74 | 0.85 | 0.90 | 0.96 | 0.99 | 1.00 | . | . | . | . | . |
| $RVol_t^{2Y}$ | −0.13 | −0.12 | −0.10 | −0.08 | −0.06 | −0.04 | 1.00 | . | . | . | . |
| $RVol_t^{3Y}$ | −0.18 | −0.15 | −0.12 | −0.09 | −0.06 | −0.02 | 0.95 | 1.00 | . | . | . |
| $RVol_t^{5Y}$ | −0.27 | −0.24 | −0.22 | −0.20 | −0.17 | −0.13 | 0.89 | 0.90 | 1.00 | . | . |
| $RVol_t^{7Y}$ | −0.33 | −0.30 | −0.28 | −0.24 | −0.21 | −0.17 | 0.81 | 0.85 | 0.97 | 1.00 | . |
| $RVol_t^{10Y}$ | −0.32 | −0.27 | −0.25 | −0.21 | −0.17 | −0.13 | 0.75 | 0.78 | 0.89 | 0.93 | 1.00 |

Unconditional Correlations of Implied and Realized Yield Volatilities

| | $RVol_t^{2Y}$ | $RVol_t^{3Y}$ | $RVol_t^{5Y}$ | $RVol_t^{7Y}$ | $RVol_t^{10Y}$ | $IV^{2Y\star}$ | $IV^{5Y}$ | $IV^{10Y}$ | MOVE |
|---|---|---|---|---|---|---|---|---|---|
| $IV^{2Y\star}$ | 0.80 | 0.75 | 0.70 | 0.60 | 0.53 | 1.00 | . | . | . |
| $IV^{5Y}$ | 0.62 | 0.61 | 0.65 | 0.66 | 0.63 | 0.79 | 1.00 | . | . |
| $IV^{10Y}$ | 0.51 | 0.52 | 0.61 | 0.65 | 0.66 | 0.59 | 0.94 | 1.00 | . |
| MOVE | 0.61 | 0.60 | 0.67 | 0.69 | 0.68 | 0.71 | 0.91 | 0.91 | 1.00 |

## II. The Model

We propose a no-arbitrage model for the joint dynamics of yields and their time-varying second moments. We introduce multiple factors in yield volatilities, and allow for priced yield and volatility risks. Although the model is cast in reduced form with latent states, it allows us to study key endogenous objects determining the term structure: term premia, expected short rates, as well as their conditional second moments, which are the focus of our empirical analysis.

Section II.A presents the baseline specification of the model. Section II.B discusses our modeling choices and how the model relates to the literature. Section II.C outlines the estimation approach. Technical details and formulas are relegated to the Internet Appendix.

### *A. The No-Arbitrage Framework: Baseline Specification*

Our framework belongs to the affine class of term structure models. We distinguish between two types of risks, which we term the yield curve ($Y_t$) and the volatility ($V_t$) factors. Their physical dynamics are given by

$$\mathrm{d}Y_t = (\mu_Y + \mathcal{K}_Y Y_t)\,\mathrm{d}t + \Sigma_Y(V_t)\mathrm{d}Z_t^{\mathbb{P}}, \tag{2}$$

$$\mathrm{d}V_t = \left(\Omega\Omega' + MV_t + V_t M'\right)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{\mathbb{P}} Q + Q'\mathrm{d}W_t^{\mathbb{P}\prime}\sqrt{V_t}, \tag{3}$$

where $Y_t$ is an $n$-dimensional vector, $V_t$ is an $m \times m$ matrix. Shocks $Z_t^{\mathbb{P}}$ and $W_t^{\mathbb{P}}$ are an $n$-dimensional vector and an $m \times m$ matrix of independent Brownian motions under the physical measure $\mathbb{P}$. Drifts are characterized by an $n$-vector $\mu_Y$, $n \times n$ matrix $\mathcal{K}_Y$, $m \times m$ matrices $M$ and $Q$ of parameters.

We depart from the affine specification of Duffie and Kan (1996) and Dai and Singleton (2000) in how we model the conditional second moments in equation (3). The $V_t$ process, introduced by Bru (1991), is a matrix-valued extension of the univariate square-root process common in finance applications (e.g., Cox, Ingersoll, and Ross (1985), Heston (1993)).[11] It offers a parsimonious representation of the variance-covariance matrix of shocks in the yield curve. To ensure that $V_t$ remains positive-definite, we impose $\Omega\Omega' = kQ'Q$, with scalar $k \geq m + 1$ (Mayerhofer, Pfaffel, and Stelzer (2011)).

In the baseline specification used in our empirical analysis, we consider three yield-curve factors summarized by $Y_t = (X_t', f_t)'$, where $X_t$ is a two-dimensional vector with dynamics

$$\mathrm{d}X_t = (\mu_X + \mathcal{K}_X X_t)\mathrm{d}t + \sqrt{V_t}\mathrm{d}Z_{X,t}^{\mathbb{P}}, \tag{4}$$

[11] The $V_t$ process, the so-called Wishart process, has been introduced to finance by Gourieroux (2006) and Gourieroux, Jasiak, and Sufana (2009). In an asset pricing context, Buraschi, Porchia, and Trojani (2010) use the Wishart process in a portfolio choice problem to study covariance hedging.

and therefore the dimension of $V_t$ is $2 \times 2$ (i.e., $m = 2$). The third yield curve state, $f_t$, has Gaussian dynamics

$$\mathrm{d} f_t = (\mu_f + \mathcal{K}_{fX} X_t + \mathcal{K}_f f_t)\mathrm{d}t + \sigma_f \mathrm{d}Z^{\mathbb{P}}_{f,t}, \tag{5}$$

where we allow $X_t$ to impact the conditional expectation of $f_t$ through the drift. The terms $\mathcal{K}_f$ and $\sigma_f$ are scalars, $\mathcal{K}_{fX}$ is a $1 \times 2$ vector, and $Z^{\mathbb{P}}_{f,t}$ is a Brownian motion independent of all other shocks in the economy. In the notation of equation (2), $\Sigma_Y(V_t)$ is a block-diagonal matrix with $\sqrt{V_t}$ and $\sigma_f$ on the diagonal. This formulation allows direct interpretation of $V_t$ as a covariance matrix of shocks to $X_t$. With $V_t$ of the form

$$V_t = \begin{pmatrix} V_{11,t} & V_{21,t} \\ V_{21,t} & V_{22,t} \end{pmatrix}, \tag{6}$$

the diagonal elements $V_{11,t}, V_{22,t}$ determine the conditional variances of the yield-curve factors, whereas $V_{21,t}$ determines their conditional covariance and has unrestricted sign.[12] The difference between (6) and the standard affine specification is that in the latter case conditional correlations between yield curve states, and thus various model-implied quantities in the yield curve, are fixed linear combinations of positive (square-root) processes. To compare the two approaches, in Section VI.C we estimate and study the implications of an $A_2(4)$ model.

Specifications (4) and (5) are empirically motivated. The combination of the dynamics of $X_t$ and $f_t$ provides scope to fit the yield curve, in line with evidence in the literature that three factors are necessary to describe the cross-section of interest rates (e.g., Litterman and Scheinkman (1991)). In terms of volatilities, we focus on matching the second moments of yields with intermediate to long maturities, whose variation we can observe with the support of high-frequency data and options. Accordingly, when the model is confronted with the data, the $X_t$ factors drive most of variation of yields with intermediate to long maturities, while $f_t$ traces the variation at the short end of the yield curve. Extensions to this setup are discussed in Section II.B and in the Internet Appendix.

To close the model, we specify the short rate and the market prices of risk. The short rate is an affine function of $X_t$ and $f_t$,

$$r_t = \gamma_0 + \gamma_X' X_t + \gamma_f f_t = \gamma_0 + \gamma_Y' Y_t, \tag{7}$$

with $\gamma_Y = (\gamma_X', \gamma_f)'$, and the stochastic discount factor has the form

$$\frac{\mathrm{d}\xi_t}{\xi_t} = -r_t \mathrm{d}t - \Lambda'_{Y,t} \mathrm{d}Z^{\mathbb{P}}_t - Tr\left(\Lambda'_{V,t} \mathrm{d}W^{\mathbb{P}}_t\right), \tag{8}$$

[12] The state $V_{21,t}$ is a "quasi-factor" in that it has its own shocks but its parameters are common with the diagonal elements of $V_t$ in a way that ensures $V_t$ is a well-defined covariance matrix. It is easiest to understand the dynamics of $V_t$ in (3) when $\Omega\Omega' = kQ'Q$ and $k$ is an integer. In that case, assuming $V_t$ of dimension $m \times m$, the $V_t$ process can be represented as a sum of $k$ outer products of $m$-dimensional Ornstein–Uhlenbeck (OU) processes with restricted drifts. In this way, the number of parameters that $V_t$ involves is not greater than the number of parameters of the underlying OU process plus one parameter for $k$. See Gourieroux (2006) and the Internet Appendix for details.

where $Tr(\cdot)$ denotes the trace operator, with the market prices of risk given as

$$\Lambda_{Y,t} = \Sigma_Y^{-1}(V_t)\left(\lambda_Y^0 + \lambda_Y^1 Y_t\right) \tag{9}$$

$$\Lambda_{V,t} = \left(\sqrt{V_t}\right)^{-1}\Lambda_V^0 + \sqrt{V_t}\Lambda_V^1. \tag{10}$$

This specification allows shocks to both yield and volatility factors to be priced, and it preserves the affine structure of the model. Therefore, we can study not only yield term premia but also variance risk premia. Equation (9) follows Duffee's (2002) essentially affine market price of risk. Equation (10) is analogous to the market price of risk proposed by Cheridito, Filipović, and Kimmel (2007, CFK) in that it allows both the constant and the mean-reversion term in the drift of volatility factors $V_t$ to change with the change of measure. While CFK apply such market price of risk to a vector of square-root processes with independent shocks and use it to model the term premium in yields, ours is an extension of this idea to the covariance matrix case, and we apply it to model the risk compensation for facing variance shocks.[13]

Prices of nominal bonds are obtained by solving $P(t,\tau) = E_t^{\mathbb{Q}}(e^{-\int_0^\tau r_s ds})$, where $\mathbb{Q}$ denotes the risk-neutral distribution. Yields, $y_t^\tau = -\frac{1}{\tau}\ln P_t^\tau$, have an affine form:

$$y_t^\tau = -\frac{1}{\tau}\{A(\tau) + B(\tau)'Y_t + Tr[C(\tau)V_t]\}. \tag{11}$$

The coefficients $A(\tau)$, $B(\tau)$, and $C(\tau)$ solve a system of ordinary differential equations. The solution for $B(\tau)$ is similar to Gaussian models while $C(\tau)$ solves a matrix Riccati equation.

The above model implies that the instantaneous expected excess return to holding a bond with maturity $\tau$, $brp_t^\tau = E_t(\frac{dP_t^\tau}{P_t^\tau}) - r_t$, is

$$brp_t^\tau = \underbrace{\left(\lambda_Y^0 + \lambda_Y^1 Y_t\right)' B(\tau)}_{brp_t^\tau|\mathrm{d}W_t^{\mathbb{P}}} + \underbrace{2Tr\left[\left(\Lambda_V^0 + V_t\Lambda_V^1\right)QC(\tau)\right]}_{brp_t^\tau|\mathrm{d}Z_t^{\mathbb{P}}}. \tag{12}$$

The notation $brp_t^\tau|\mathrm{d}W_t^{\mathbb{P}}$ ($brp_t^\tau|\mathrm{d}Z_t^{\mathbb{P}}$) is shorthand for the instantaneous expected excess bond return conditional on a $\mathrm{d}W_t^{\mathbb{P}}$ ($\mathrm{d}Z_t^{\mathbb{P}}$) shock. Thus, the first term on the right-hand side in (12) denotes the instantaneous expected excess return for facing shocks to yield curve states, $\mathrm{d}Z_t^{\mathbb{P}}$, and is common in Gaussian term structure models. The second term is the instantaneous expected excess return for facing shocks to volatility states, $\mathrm{d}W_t^{\mathbb{P}}$, and is a consequence of priced

[13] To ensure that (10) does not admit arbitrage opportunities, we impose $\Lambda_V^0 = vQ'$, for scalar $v$ such that $k - 2v \geq m + 1$, which guarantees boundary nonattainment for $V_t$ under the risk-neutral measure, $\mathbb{Q}$ (see Mayerhofer (2014) for details). This is analogous to the condition that CFK provide for $A_m(n)$ affine models, in which volatility dynamics are driven by a vector of square-root processes with independent shocks. CFK show that boundary nonattainment is required to exclude arbitrage opportunities in such a setting.

volatility risk in our model. Thus, the volatility states are allowed to affect both the levels of yields in equation (11) and bond risk premia in equation (12).

The instantaneous yield covariation is driven only by the $V_t$ states

$$
\begin{aligned}
v_t^{\tau_i,\tau_j} &:= \frac{1}{\mathrm{d}t}\langle \mathrm{d}y_t^{\tau_i}, \mathrm{d}y_t^{\tau_j}\rangle \\
&= \frac{1}{\tau_i\tau_j}\left\{Tr\{[B_X(\tau_i)B_X(\tau_j)' + 4C(\tau_i)Q'QC(\tau_j)]V_t\} + B_f(\tau_i)B_f(\tau_j)\sigma_f^2\right\}, \quad (13)
\end{aligned}
$$

where we define $B(\tau) = (B_X(\tau)', B_f(\tau))'$ for coefficients associated with $X_t$ and $f_t$. The conditional risk-neutral expectation of the annualized yield variance over horizon $h$ (measured as fraction of a year) is

$$
v_{t,t+h}^{\mathbb{Q},\tau} = \frac{1}{h}E_t^{\mathbb{Q}}\int_0^h v_{t+s}^{\tau}\mathrm{d}s, \tag{14}
$$

which, similar to (13), is an affine function of $V_t$ and is available in closed form. Analogously, $v_{t,t+h}^{\mathbb{P},\tau}$ is the expected $h$-period variance under the physical dynamics. The variance risk premium, defined as the difference between the expected $h$-period variance under the physical and risk-neutral measure, is

$$
vrp_{t,t+h}^{\tau} = v_{t,t+h}^{\mathbb{Q},\tau} - v_{t,t+h}^{\mathbb{P},\tau}, \tag{15}
$$

and is determined by the volatility states $V_t$ and the corresponding market price of risk parameters in equation (10).

## *B. Discussion*

The fact that $V_t$ appears in expression (11) distinguishes our model from the USV settings, which impose explicit restrictions so that volatility factors do not affect the cross-section of yields. Such a separation usually improves the volatility fit of low-dimensional ATSMs (Collin-Dufresne, Goldstein, and Jones (2009)). However, as highlighted by Joslin (2014), there are few reasons except statistical ones for this constraint to strictly hold in the data. We leave it to the data to decide how sensitive the cross-section of yields is to volatility.

Our approach to modeling yield volatilities also differs from Buraschi, Cieslak, and Trojani (2010, BCT). BCT also use process (3) but they assume that it drives the yield curve factors. Since this leads to yield volatilities and levels being different linear combinations of the *same* state variables, their model shares some of the problems typical of the low-dimensional ATSMs. In contrast, we use process (3) to describe the covariance matrix of shocks to the yield curve factors. This specification is crucial for the empirical fit and thus for our interpretation of yield volatility components in that it combines the convenience for second moments with the flexibility that Gaussian dynamics have for describing the term premia.

One could consider extending the baseline specifications in (4) and (5) in at least two ways. First, we could allow shocks to all yield curve states in

$Y_t$ to be linked by a stochastic covariance matrix, in which case $V_t$ would be of dimension $3 \times 3$, with three volatility states and three covariance states. Second, we could estimate a model that maintains the current $2 \times 2$ structure of $V_t$ but allows $V_t$ to affect the conditional volatility of $f_t$. The analysis of the cross-sectional and dynamic fit, provided in Section III, indicates that our current volatility specification has sufficient flexibility to capture the important features of the conditional second moments of yields. Thus, estimation of a larger and likely overparameterized $3 \times 3$ model for the volatility dynamics does not seem warranted by the data. We consider the second extension in the Internet Appendix and find that it does not result in a better fit to volatilities relative to our baseline case.

### C. Estimation Approach

We estimate the model on a weekly frequency ($\Delta t = \frac{1}{52}$) combining pseudo-maximum likelihood with filtering. Details of the estimation approach, identification, as well as parameter estimates are presented in the Internet Appendix.

We introduce four types of measurements: zero-coupon yields ($\tilde{y}_t^\tau$), conditional expectations of one-year-ahead yields from surveys ($\tilde{E}_t^s(y_{t+h_s}^\tau), h_s = 1$), quadratic co-variation of yields ($\tilde{v}_t^{\tau_i,\tau_j}$), and the expected risk-neutral variance of yields over the next month ($\tilde{v}_{t,t+h_v}^{\mathbb{Q},\tau}, h_v = \frac{1}{12}$). Notation with a tilde distinguishes measurements used in estimation from the model-implied quantities. Measurements are observed with errors, which we assume to be mutually independent and serially uncorrelated.

In estimation, we use six yields with maturities of six months and two, three, five, seven, and 10 years. To facilitate estimation of the yield dynamics under the physical measure, we include survey expectations of interest rates from the Blue Chip Financial Forecasts (BCFF). Specifically, we use the one-year-ahead median forecast of the six-month and the two-year yield.[14] Interpretation of interest rate volatility, which we pursue below, depends in part on how reliably we are able to identify the term premia and short-rate expectations components of the yield curve. Kim and Orphanides (2012) show in a Gaussian setting that inclusion of survey expectations of interest rates significantly increases the precision relative to a yields-only estimation.

In terms of second-moment measurements, $\tilde{v}_t^{\tau_i,\tau_j}$ is obtained from the high-frequency zero-coupon yield curve using the realized covariance estimator (1). The expected risk-neutral variance $\tilde{v}_{t,t+h_v}^{\mathbb{Q},\tau}$ is the squared one-month-ahead implied volatility series.[15] We include four realized variance measurements,

[14] Panelists in the BCFF survey provide forecasts of constant-maturity Treasury yields, which are par coupon yields. We convert them to a zero-coupon basis. The Internet Appendix provides details on the BCFF survey.

[15] Since options used to obtain implied variances have futures on a hypothetical coupon bond as underlying, we use the average duration of the bond during our sample period to approximate the zero-coupon implied volatility. The implied volatility measurements are matched with the risk-neutral volatility from the model, $v_{t,t+h_s}^{\mathbb{Q},\tau}$, with $\tau = 1.9$, 4.4, and 7.5 years. These numbers represent the average duration of the coupon bond underlying the interest rate futures in our sample.

namely, variances of the two-, five-, and 10-year yield and the covariance between the five- and 10-year yield, as well as three measurements for the implied variance with underlying bond maturities of two, five, and 10 years. Since the implied variance series for the two-year maturity are available from March 2006, in estimation we treat the initial observations as missing. In total, we have seven measurements to pin down the physical and risk-neutral parameters of $V_t$.

Yields as well as implied and realized variances are measured on the last business day of a week. Interest rate surveys are compiled every month during the final week of the month and are published on the first day of the subsequent month. Thus, we line up surveys with other measurements as of the last week of a month, and treat the remaining observations as missing values.

To address the non-Gaussianity in factor dynamics, we use the square-root unscented Kalman filter (UKF) proposed by Julier and Uhlmann (1997). We maximize the likelihood with differential evolution, which is a global optimization algorithm (Price, Storn, and Lampinen (2005)). After imposing the econometric identification restrictions the model has 29 parameters, of which 19 are associated with the yield curve factors $Y_t$ and 10 are associated with the volatility factors $V_t$.[16]

## III. Empirical Performance of the Model

As a prerequisite for an interpretation of yield volatility, this section studies the model's empirical performance in terms of the cross-sectional and dynamic fit. The results can be summarized as follows: First, the model captures the persistent variation of yield variances by filtering out the jagged high-frequency dynamics of the realized second moments. Second, we observe no apparent tension in matching yield variances at different maturities, or in matching the physical and risk-neutral volatility dynamics. Third, the ability to fit the conditional second moments does not come at the cost of a poor fit to the cross section of yields or to the conditional expectations from surveys. In terms of bond risk premia, the model has flexibility similar to that in a Gaussian setting.

### *A. Fit to the Conditional Second Moments of Yields*

#### *A.1. Cross-Sectional Fit*

The cross-sectional fit of the model is summarized in Table II and Figure 2. In Figure 2, we superimpose the realized and implied variances and the realized covariances with their model-based counterparts. The realized second moments have distinct high-frequency dynamics, which in the graph

[16] Following prior literature (e.g., Duffee (2002), Joslin, Le, and Singleton (2013)), we restrict three insignificant parameters in the mean-reversion matrix of the $Y_t$ dynamics to zero. We find these restrictions to be essentially inconsequential for the dynamics of short-rate expectations and term premia and their respective volatilities. Details are discussed in the Internet Appendix.

**Table II**

## Fit of the Baseline Model to the Second Moments of Yields

Panel A summarizes the cross-sectional fit to realized variances, realized covariances, and implied variances by the baseline model. Rows (1) and (2) report the RMSEs for realized variances ($\times 10^4$) and realized volatilities (in basis points). For the few data points where covariances become negative (seven observations in total across all measurements), to compute the RMSE in row (2) we take the square root of the absolute value of the measurement. Rows (3) though (5) are based on regression (16). Row (3) reports the $p$-values testing the null hypothesis that the regression intercept is zero $\alpha_0 = 0$; row (4) shows $p$-values for the Wald test of the null that all regression coefficients are zero $(\alpha_0, \alpha_1')' = 0$. * indicates rejection of the null at the 5% level. The Wald test does not use any adjustment of the parameter covariance matrix. Row (5) provides the adjusted $\bar{R}^2$. Panel B reports regressions of the model-implied variance forecast errors specified in equation (17). The forecast errors are computed for horizons $h$ of one, four, and eight weeks ahead. Row (1) reports $p$-values testing the null that the regression intercept is zero $\beta_0 = 0$; row (2) shows $p$-values for the Wald test of the null that all regression coefficients are zero $(\beta_0, \beta_1')' = 0$. * indicates rejection of the null at the 5% level. The Wald test is based on Newey–West adjustment with $h - 1$ lags. Row (3) reports the adjusted $\bar{R}^2$.

Panel A. Cross-Sectional Fit

| | Realized Variance | | | | | Realized Covariance | | | Implied Variance | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2Y | 3Y | 5Y | 7Y | 10Y | 2,5Y | 2,10Y | 5,10Y | 5Y | 10Y |
| (1) RMSE Var $\times 10^4$ | 0.74 | 0.92 | 0.74 | 0.58 | 0.61 | 0.67 | 0.44 | 0.43 | 0.29 | 0.22 |
| (2) RMSE Vol (bps) | 24.96 | 26.49 | 22.74 | 20.70 | 20.99 | 25.56 | 20.60 | 18.58 | 10.61 | 9.47 |
| (3) $p$-value ($\alpha_0 = 0$) | 0.61 | 0.29 | 0.94 | 0.93 | 0.39 | 0.08 | 0.29 | 0.00* | 0.06 | 0.08 |
| (4) $p$-value $(\alpha_0, \alpha_1')' = 0$ | 0.36 | 0.21 | 0.44 | 0.42 | 0.49 | 0.01* | 0.09 | 0.02* | 0.00* | 0.01* |
| (5) $\bar{R}^2$ | 0.02 | 0.00 | 0.01 | 0.01 | 0.03 | 0.07 | 0.18 | 0.06 | 0.14 | 0.06 |

Panel B. Dynamic Fit

| | Realized Variance | | | | | Realized Covariance | | |
|---|---|---|---|---|---|---|---|---|
| | 2Y | 3Y | 5Y | 7Y | 10Y | 2,5Y | 2,10Y | 5,10Y |
| $h = 1$ week | | | | | | | | |
| (1) $p$-value $\beta_0 = 0$ | 0.93 | 0.31 | 0.99 | 0.87 | 0.61 | 0.28 | 0.36 | 0.00* |
| (2) $p$-value $(\beta_0, \beta_1')' = 0$ | 0.05* | 0.06 | 0.01* | 0.01* | 0.00* | 0.13 | 0.00* | 0.03* |
| (3) $\bar{R}^2$ | 0.03 | 0.03 | 0.03 | 0.02 | 0.02 | 0.07 | 0.15 | 0.07 |
| $h = 4$ weeks | | | | | | | | |
| (1) $p$-value $\beta_0 = 0$ | 0.99 | 0.45 | 0.77 | 0.42 | 0.75 | 0.44 | 0.51 | 0.01* |
| (2) $p$-value $(\beta_0, \beta_1')' = 0$ | 0.67 | 0.74 | 0.71 | 0.96 | 0.27 | 0.99 | 0.46 | 0.65 |
| (3) $\bar{R}^2$ | 0.07 | 0.07 | 0.05 | 0.04 | 0.03 | 0.11 | 0.18 | 0.10 |
| $h = 8$ weeks | | | | | | | | |
| (1) $p$-value $\beta_0 = 0$ | 0.78 | 0.47 | 0.70 | 0.28 | 0.36 | 0.75 | 0.82 | 0.05* |
| (2) $p$-value $(\beta_0, \beta_1')' = 0$ | 0.85 | 0.81 | 0.97 | 0.81 | 0.22 | 0.80 | 0.53 | 0.73 |
| (3) $\bar{R}^2$ | 0.08 | 0.07 | 0.06 | 0.05 | 0.05 | 0.11 | 0.18 | 0.11 |

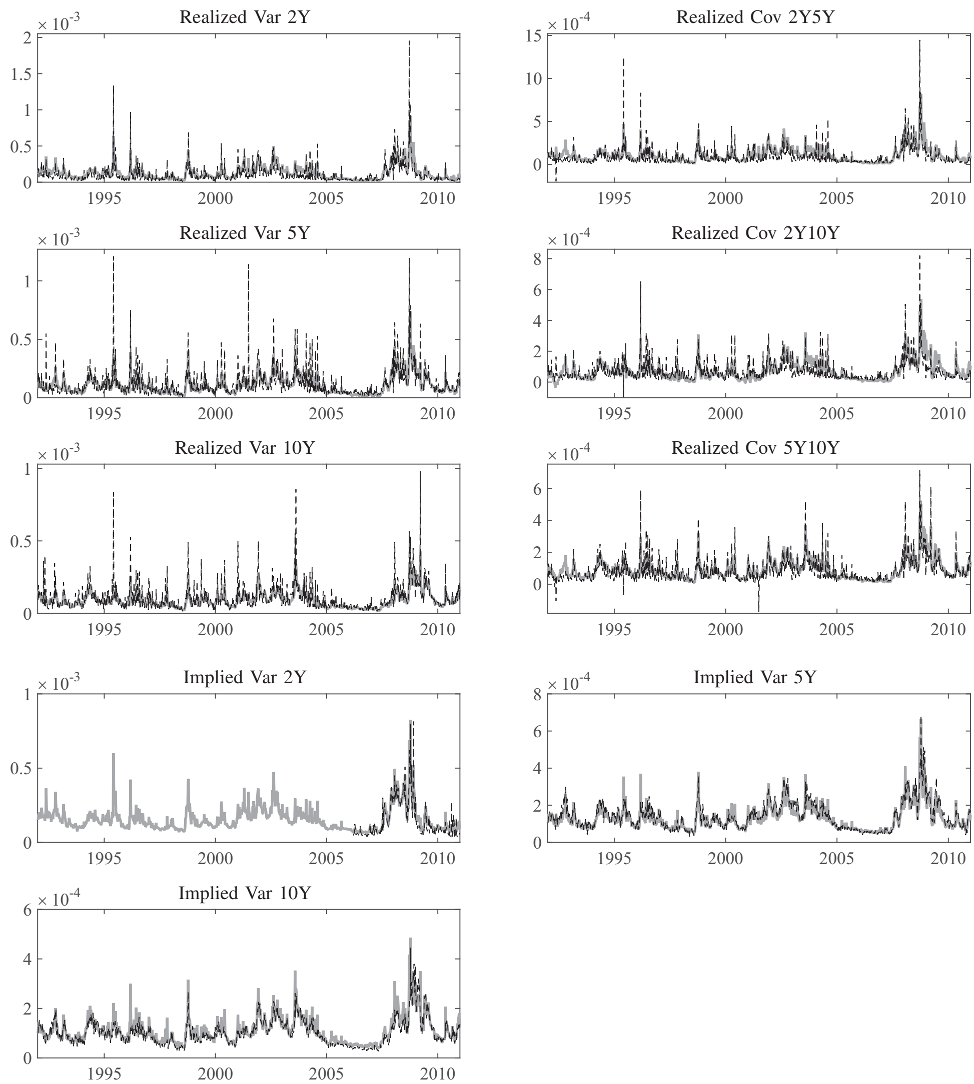


**Figure 2. Fit of the baseline model to the realized and implied second moments of yields.** The figure presents the fit to the realized yield variances, realized covariances, and implied variances for maturities of two, five, and 10 years obtained from the baseline model. The solid line is the fitted value, while the dashed line corresponds to the data. The data for the implied variance at the two-year maturity start in 2006; before this date, they are treated as missing observations in estimation.

reveals itself through large and short-lived spikes. The yield variance $v_t^\tau$ filtered by the model is smoother than the realized variance in the data. This is because the model focuses on the ex-ante expectations of volatility and the isolated large positive outliers typical of the realized variance should not be predictable a priori (e.g., Andersen, Bollerslev, and Meddahi (2005)).

In Panel A of Table II (rows 1 and 2), we report the root mean squared errors (RMSEs) for different volatility measurements. The RMSEs for the implied volatilities are about 10 basis points; those for realized volatilities are about 10 basis points higher as the realized volatility is significantly more noisy. These results suggest that the model is able to filter through the measurement error and other features of the data (e.g., jumps) that are absent in our specification.

To evaluate the cross-sectional performance more formally, we regress the fitting errors on lagged information,

$$\mathcal{M}_t - \widehat{\mathcal{M}}_t^{\text{mod}} = \alpha_0 + \sum_{i=1}^{3} \alpha_{1,i}^{\text{yld}} PC_{i,t-\Delta t}^{\text{yld}} + \sum_{i=1}^{3} \alpha_{1,i}^{\text{var}} PC_{i,t-\Delta t}^{\text{var}} + \varepsilon_t, \tag{16}$$

where $\mathcal{M}_t$ is a second-moment measurement observed at time $t$, $\widehat{\mathcal{M}}_t^{\text{mod}}$ is its model-implied counterpart, and $PC_{i,t-\Delta t}^{\text{yld}}$ and $PC_{i,t-\Delta t}^{\text{var}}$ are the principal components (PCs) of yields and variances, respectively, where in constructing the latter we include both the realized and the implied second moments. Time is measured in weeks, that is, $t - \Delta t$ denotes information lagged by one week.

In Panel A of Table II (rows 3 and 4), we report $p$-values from the Wald test of the null hypotheses that the intercept is zero, $\alpha_0 = 0$, and that all regression coefficients are zero, $(\alpha_0, \alpha_1')' = 0$. Under the null, the residual in the regression is serially uncorrelated and thus we use simple unadjusted standard errors. This approach is conservative in that it makes it more likely for the test to reject, and to suggest model misspecification. For most volatility measurements, we cannot reject either of the nulls as $p$-values exceed the 5% level. The last row of Panel A gives the adjusted $R^2$s ($\bar{R}^2$s) from the regressions. The $\bar{R}^2$s are low (below 3%) for the realized variances and slightly higher for the realized covariances and the implied variances. The highest $R^2$ of 18% is obtained for the realized covariance between the two- and 10-year yields, even though, based on Figure 2, its fit is not much worse than for other volatility measurements.

Regression (16) helps assess whether there are aspects of yield volatility dynamics that our model systematically misses. Such a test puts a relatively high bar on the model. First, the three volatility states that we obtain from a fully specified term structure model are different from the PCs used in (16) that summarize purely cross-sectional information in volatilities. Second, the model does not admit a conditional dependence of yield volatility on the level of yields,[17] and thus the presence of lagged PCs of yields, $PC^{\text{yld}}$, in (16) is outside the model's scope. Since we lack a frame of reference to evaluate such regressions in the context of interest rate volatility, we discuss two benchmarks for their interpretation. First, we note that regressions analogous to (16) on yield fitting errors in a standard three-factor Gaussian model also reject the null (at

[17] This feature is not particular to our model, and arises due to admissibility constraints in the general class of affine models with stochastic volatility.

least for some maturities) and can have $R^2$ above 10%.[18] By this comparison, the cross-sectional fit of our model to yield volatility is comparable to the fit to yield levels that is common in the literature. Second, in Section VI.C we report estimates of regression (16) for the volatility fitting errors implied by the $A_2(4)$ model, and find that the null hypothesis is more frequently rejected than in our baseline setting.

### *A.2. Dynamic Fit*

We also study the properties of volatility forecast errors. We define the forecast error as $\mathcal{M}_{t+h\Delta t} - E_t^{\mathrm{mod}}(\mathcal{M}_{t+h\Delta t})$, where $\mathcal{M}_{t+h\Delta t}$ is the realized variance or covariance observed at time $t + h\Delta t$ and $E_t^{\mathrm{mod}}(\mathcal{M}_{t+h\Delta t})$ is the corresponding model-based expectation of the variance or covariance $h$ periods ahead. We estimate the following regressions:

$$\mathcal{M}_{t+h\Delta t} - E_t^{\mathrm{mod}}(\mathcal{M}_{t+h\Delta t}) = \beta_0 + \sum_{i=1}^{3} \beta_{1,i}^{\mathrm{yld}} PC_{i,t}^{\mathrm{yld}} + \sum_{i=1}^{3} \beta_{1,i}^{\mathrm{var}} PC_{i,t}^{\mathrm{var}} + \varepsilon_{t+h\Delta t}. \quad (17)$$

Under the null hypothesis that the model is correctly specified, the regression coefficients are zero but the residual is autocorrelated due to overlapping observations. Therefore, we use Newey–West standard errors with $h-1$ lags. We report the $p$-values for the Wald test of the restriction that the intercept is zero, $\beta_0 = 0$, and that all regression coefficients are jointly zero, $(\beta_0, \beta_1')' = 0$. Panel B of Table II reports the results for three forecast horizons ($h = 1, 4$ and 8 weeks), and for realized variances and covariances. The test does not reject the zero intercept restriction except in one case. The rejection of the joint zero restriction on all coefficients occurs for several measurements at the shortest horizon of one week, but not at longer horizons.

## *B. Fit to the First Moments of Yields*

We verify that the ability of the model to match the second moments is not traded off against the fit to the first moments of yields. Our estimation imposes the requirement that the model matches not only yields but also physical expectations of interest rates from surveys. Table III, Panel A shows that the RMSEs are about five basis points on average for yields across maturities, and 18 basis points for survey expectations. We provide analogous RMSEs for a three-factor Gaussian model that we estimate on the same yield and survey measurements as our baseline model. The fit from the two models is comparable.

Next, we study whether our model gives rise to empirically plausible expected returns and term premia. We compare the model's implications to three alternative risk premium proxies from the literature: (i) the Cochrane and Piazzesi

[18] While the RMSEs of yields are about five basis points on average across maturities, when we regress yield fitting errors on lagged PCs as in (16), the null hypothesis that all coefficients are zero is rejected at several maturities, and the $R^2$ from the regression can exceed 10%. These regressions can be found in the Internet Appendix.

**Table III**

**Fit of the Baseline Model to the First Moments of Yields**

Panel A provides the RMSEs in basis points for the fit to yields and to survey expectations of yields. Row (1) is based on our baseline model; row (2) is based on a three-factor Gaussian model estimated with the same yield and survey data as the baseline. Panel B reports regressions of the model-implied expected returns on alternative risk premium proxies: the Cochrane-Piazzesi (CP) factor, the Federal Reserve Board (FRB) term premium, and the survey-based one-year expected excess return for the two-year bond (BCFF2Y). In the column "CP," the LHS variable is the average (across maturities) model-implied expected excess return; LHS and RHS variables are standardized and therefore the intercept is not reported. In the column "FRB," we regress the average instantaneous term premium in our model on the corresponding average instantaneous FRB term premium, where averages are taken across maturities from two to 10 years. In the column "BCFF 2Y" we use the instantaneous bond risk premium on the LHS; LHS and RHS are expressed in annual returns units, for example, an intercept of 0.002 equals 20 basis points. The data span the period 1992:01 to 2010:12. Due to availability of surveys, all series in Panel B are sampled at the monthly frequency. We use the last week of the month to convert weekly model-implied quantities to the monthly frequency. *t*-statistics in parentheses are Newey–West adjusted with 18 lags.

Panel A. Cross-Sectional Fit

| | Yields | | | | | | Survey Expectations | |
|---|---|---|---|---|---|---|---|---|
| RMSE (bps) | 6M | 2Y | 3Y | 5Y | 7Y | 10Y | $E_t^s(y_{t+1Y}^{(6M)})$ | $E_t^s(y_{t+1Y}^{(2Y)})$ |
| (1) Our baseline | 5.33 | 6.78 | 6.04 | 4.71 | 3.27 | 5.30 | 18.10 | 18.37 |
| (2) 3f Gaussian | 7.35 | 6.46 | 4.85 | 4.25 | 3.12 | 4.36 | 21.72 | 24.34 |

Panel B. Comparison of Risk Premia from the Baseline Model with Alternative Proxies

| Regressor: | CP | FRB | BCFF2Y |
|---|---|---|---|
| $\beta_0$ | – | −0.004 | 0.002 |
| | – | (−10.03) | (2.83) |
| $\beta_1$ | 0.72 | 1.07 | 1.02 |
| | (6.37) | (26.01) | (9.19) |
| $\bar{R}^2$ | 0.52 | 0.95 | 0.66 |

(2005, CP) factor, (ii) the term premium published by the Federal Reserve Board (FRB), and (iii) the survey-based risk premium. The CP factor is obtained following Cochrane and Piazzesi (2006). The FRB premium is obtained from the FRB's website and is based on a three-factor Gaussian model estimated using yields and survey expectations of interest rates as measurements. It therefore serves as a check on whether we are able to replicate the success of Gaussian models in generating economically meaningful risk premia. The survey-based premium is obtained directly from the BCFF survey of the one year yield one -year ahead; this survey forecast is not used in the estimation of our model. For consistency across the different proxies, we sample the data at the month's end.[19]

[19] The CP factor is estimated over the 1971 to 2011 period using the Gürkaynak, Sack, and Wright (2006) zero-coupon yields, which Cochrane and Piazzesi (2008) use. The FRB term premia

Table III, Panel B, presents regressions of the model-implied premia on the three measures. A regression on the survey-based premium generates a highly significant slope coefficient of 1.02 and an $R^2$ of 66%. Similarly, the CP factor explains more than 50% of the risk premium variation in the model. With a slope coefficient of 1.07 and an $R^2$ reaching 95%, the closest alignment is between the term premium in our model and the FRB term premium. This is despite the fact that our model admits a richer form of market prices of risk (equation (12)) than the purely Gaussian model that underlies the FRB premium. We discuss the economic interpretation of this finding in Section V.

## IV. Decomposing Conditional Yield Volatility

Yield volatility stems from either volatile short-rate expectations or volatile term premia and the comovement between the two (plus a convexity term). In this section, we use our model to analyze the properties of these components. We then study the effect that stochastic volatility has on the conditional distribution of interest rates. We also characterize the conditional correlation between shocks to term premia and shocks to short-rate expectations.

### *A. Volatility of Term Premia versus Volatility of Short-Rate Expectations*

A yield with maturity $\tau$ can be decomposed as

$$y_t^{\tau} = y_t^{E,\tau} + y_t^{TP,\tau} + y_t^{C,\tau}, \tag{18}$$

where

$$y_t^{E,\tau} = \frac{1}{\tau} E_t^{\mathbb{P}} \left( \int_0^{\tau} r_{t+s} \right) \mathrm{d}s \tag{19}$$

$$y_t^{TP,\tau} = \frac{1}{\tau} \left[ E_t^{\mathbb{Q}} \left( \int_0^{\tau} r_{t+s} \right) \mathrm{d}s - E_t^{\mathbb{P}} \left( \int_0^{\tau} r_{t+s} \right) \mathrm{d}s \right] \tag{20}$$

$$y_t^{C,\tau} = -\frac{1}{\tau} \left[ \ln E_t^{\mathbb{Q}} \left( \exp \left( - \int_0^{\tau} r_{t+s} \mathrm{d}s \right) \right) + E_t^{\mathbb{Q}} \left( \int_0^{\tau} r_{t+s} \mathrm{d}s \right) \right], \tag{21}$$

$y_t^{E,\tau}$ is the expected average short rate over the life of a $\tau$-period bond, $y_t^{TP,\tau}$ is the term premium, and $y_t^{C,\tau}$ is the convexity. We focus on the conditional second moments of the first two terms, which constitute the main portion of

are based on the model of Kim and Wright (2005). Compared to our estimation, Kim and Wright (2005) use different maturities and horizons of interest rate surveys. For the survey-based proxy, we compute $E_t^s(rx_{t+1Y}^{(2)}) = f_t^{(2)} - E_t^s(y_{t+1Y}^{(1)})$, where $f_t^{(2)}$ is a one-year-forward rate locked in today for a loan starting in a year, and $E_t^s(y_{t+1Y}^{(1)})$ is the one-year-ahead expectation (median forecast) of the one-year zero-coupon yield backed out from the BCFF survey. Note that we do not use the $E_t^s(y_{t+1Y}^{(1)})$ survey forecast in the estimation of our model.

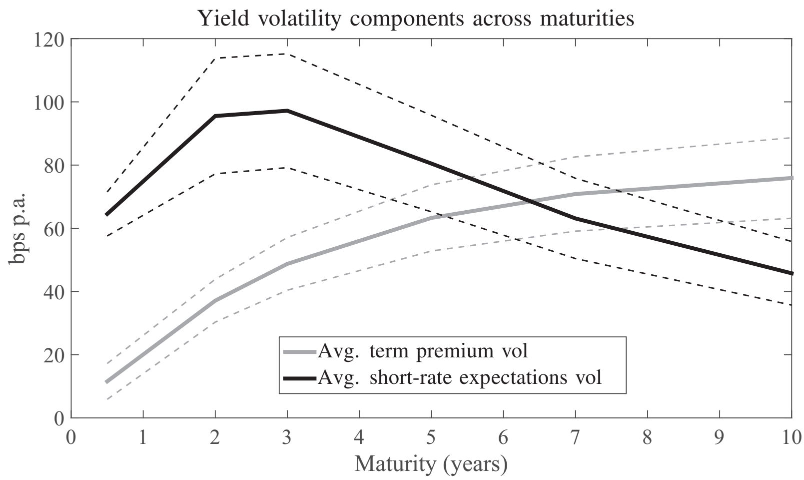


**Figure 3. Average conditional volatilities of short-rate expectations and the term premium.** The figure displays the term structure of sample average conditional volatilities (thick line) of expected short rates and of term premia. The asymptotic 95% confidence intervals are obtained using the delta method (dashed lines).

the total conditional yield volatility. Specifically, we compute the corresponding instantaneous variances and the covariance according to

$$v_t^{E,\tau} = \frac{1}{\mathrm{d}t}\langle \mathrm{d}y_t^{E,\tau}\rangle, \quad v_t^{TP,\tau} = \frac{1}{\mathrm{d}t}\langle \mathrm{d}y_t^{TP,\tau}\rangle, \quad v_t^{E,TP,\tau} = \frac{1}{\mathrm{d}t}\langle \mathrm{d}y_t^{E,\tau}, \mathrm{d}y_t^{TP,\tau}\rangle. \tag{22}$$

Next, using the orthogonality between $\mathrm{d}Z_t^{\mathbb{P}}$ and $\mathrm{d}W_t^{\mathbb{P}}$, the total instantaneous yield variance can be written as

$$v_t^{\tau} = v_t^{E,\tau} + v_t^{TP,\tau} + 2v_t^{E,TP,\tau} + v_t^{C,\tau}, \tag{23}$$

where the last term is the volatility-of-volatility, or "vol-of-vol," effect. Expression (23) is a different way to view the instantaneous yield variance in (13). Each element in (23) is affine in $V_t$, and thus has a tractable form (provided in the Internet Appendix). In Panel A of Table IV, we compute the average contribution of each term to the total variance. The first three terms account for essentially all of the yield variance across maturities. In the discussion that follows, we convert variances into volatilities expressed in basis points per annum.

### *A.1. Behavior of Volatility Components Across Maturities*

Using the decomposition (22), Figure 3 traces out the sample averages of the instantaneous volatilities of short-rate expectations and of term premia across maturities. The dashed lines are the asymptotic 95% confidence intervals. Short-rate expectations volatility peaks at a maturity between two and three years and declines at longer maturities. In contrast, the average term

**Table IV**

**Decomposition of the Conditional Yield Volatility**

Panel A reports the sample averages of the ratio of short-rate expectations variance to total variance, $v_t^{E,\tau}/v_t^{\tau}$, the term premium variance to total variance, $v_t^{TP,\tau}/v_t^{\tau}$, and twice the covariance between the premium and expectations to total variance, $2v_t^{E,TP,\tau}/v_t^{\tau}$. These components are defined in equations (22) and (23). Row (4) is the sum of the three ratios. The sum can be below 100 due to a nonzero contribution of $v_t^{C,\tau}$; this component is not reported separately. Panel B summarizes the sample properties of the volatilities of short-rate expectations and the term premia, $\sqrt{v_t^{E,\tau}}$ and $\sqrt{v_t^{TP,\tau}}$, expressed in basis points per annum. The 95% confidence intervals for the average sample volatility are reported in row (2) and are constructed using the delta method.

| Panel A. Decomposition of Average Conditional Yield Variance (in %) | | | | | | |
|---|---|---|---|---|---|---|
| | 6M | 2Y | 3Y | 5Y | 7Y | 10Y |
| (1) Mean($v_t^{E,\tau}/v_t^{\tau}$) | 80.68 | 85.03 | 86.75 | 75.26 | 53.28 | 29.20 |
| (2) Mean($v_t^{TP,\tau}/v_t^{\tau}$) | 2.52 | 15.31 | 25.99 | 51.47 | 70.01 | 78.32 |
| (3) Mean($2v_t^{E,TP,\tau}/v_t^{\tau}$) | 16.81 | −0.34 | −12.75 | −26.80 | −23.42 | −7.75 |
| (4)=(1)+(2)+(3) | 100.00 | 100.00 | 99.98 | 99.93 | 99.86 | 99.77 |
| **Panel B. Properties of Term Premium and Short-Rate Expectations Volatility** | | | | | | |
| Volatility of Short-Rate Expectations | | | | | | |
| (1) Mean (bps p.a.) | 64.53 | 95.53 | 97.19 | 80.47 | 63.10 | 45.75 |
| (2) 95% CI | [57.6; 71.5] | [77.2; 113.8] | [79.2; 115.2] | [65.2; 95.7] | [50.4; 75.8] | [35.7; 55.8] |
| (3) St.dev (bps p.a.) | 6.73 | 34.05 | 37.19 | 31.68 | 24.75 | 17.59 |
| (4) Half-life (weeks) | 6.84 | 7.80 | 8.37 | 8.34 | 7.91 | 7.15 |
| Volatility of Term Premia | | | | | | |
| (1) Mean (bps p.a.) | 11.55 | 37.10 | 48.74 | 63.28 | 70.84 | 75.90 |
| (2) 95% CI | [5.9; 17.2] | [30.3; 43.9] | [40.4; 57.1] | [52.8; 73.8] | [59.1; 82.6] | [63.2; 88.7] |
| (3) St.dev (bps p.a.) | 3.19 | 10.43 | 14.04 | 18.69 | 21.12 | 22.76 |
| (4) Half-life (weeks) | 8.31 | 15.07 | 18.70 | 22.60 | 24.23 | 25.26 |

premium volatility increases with maturity. Other sample properties are summarized in Panel B of Table IV.

The cross-sectional behavior of average volatilities is economically meaningful. The hump in volatility of short-rate expectations confirms the view shared by practitioners and policy makers that the two- to three-year maturity segment is highly sensitive to market expectations about the future short rate and, by extension, about the path of monetary policy (e.g., Stein (2013)). An increasing premium volatility as a function of maturity instead supports the intuition that longer-maturity yields are more informative about the term premia than the short end of the yield curve.

The above conclusions are useful to the extent that they are not overwhelmed by the uncertainty surrounding model estimates. Such uncertainty can be large in term structure applications, especially when it comes to parameters governing the physical dynamics of interest rates estimated with short samples. Consistent with Kim and Orphanides (2012), however, we find that inclusion of interest rate survey forecasts significantly increases the precision with which we can separate risk premia from short-rate expectations, and therefore their corresponding volatilities.

### *A.2. Dynamic Features of the Volatility Components*

The left-hand panels of Figure 4 present the time series of $\sqrt{v_t^{TP,\tau}}$ and $\sqrt{v_t^{E,\tau}}$ obtained using equation (22) for maturities of two, five, and 10 years. The right-hand panels plot an analogous decomposition of the level of interest rates into term premium and average expected short rate using equations (19) and (20). The graph visually highlights the point that at shorter maturities the conditional volatility is dominated by the short-rate expectations component, while at long maturities it is dominated by the term premium component, even though the volatility of short-rate expectations is still quantitatively important at the long end of the yield curve. The volatility of term premia is more persistent than the volatility of short-rate expectations, with their respective sample half-lives equal to 21 and 8 weeks on average for maturities between two and 10 years. Term premia and short-rate volatility components comove relatively weakly with each other, with a sample correlation not exceeding 0.25 at any given maturity, but each features strong comovement across maturities. Such behavior of volatilities is consistent with the decomposition of yield levels, where the right-hand panels of Figure 4 illustrate that term premia and short-rate expectations each move on one dominant factor across maturities.[20]

[20] It is useful to compare this decomposition to the one based on the PCs. It is well known that the first PC (the level factor) explains more than 98% of the variation in yields across maturities. In terms of economic drivers, however, the first PC combines both term premia and short-rate expectations, which need not be highly correlated (based on our model, the correlation between the two terms is about 0.5 in levels and 0.2 in monthly changes on average across maturities). Similar intuition pertains to yield volatilities: while the first PC extracted from realized and implied volatilities explains about 80% of their variation (Section I.D), our decomposition shows that it is a combination of economically different volatility components.

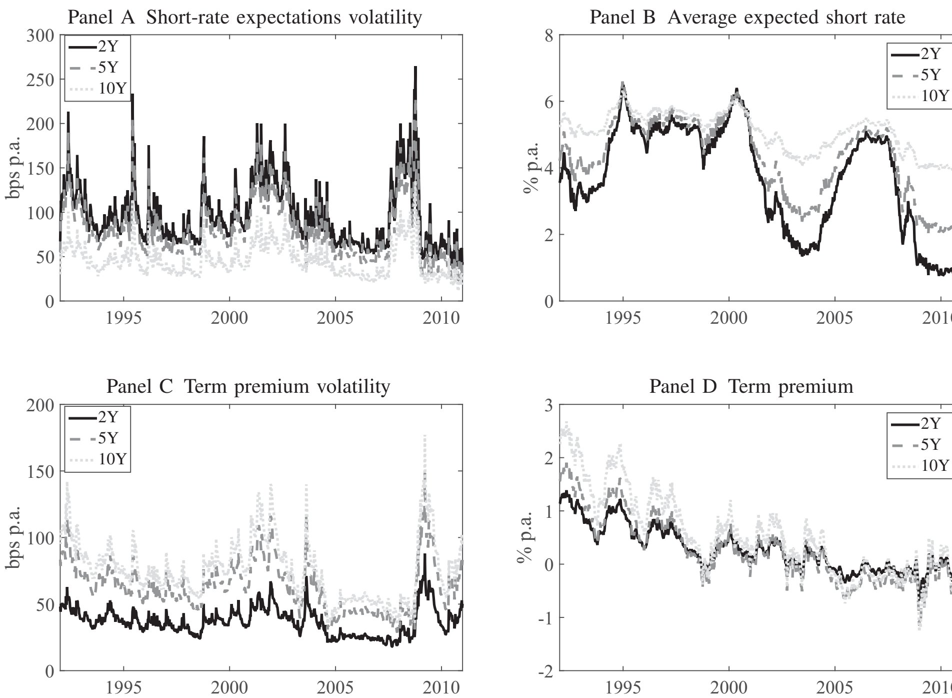


**Figure 4. Time series of term premia, short-rate expectations, and their conditional volatilities.** The top panels show the time series of short-rate expectations volatility (Panel A) and the average expected short rate (Panel B). The bottom panels present analogous dynamics for the volatility of term premium (Panel C) and the level of the term premium (Panel D). All plots report maturities of two, five, and 10 years.

It is informative to study the behavior of conditional volatilities over the business cycle. Figure 5 juxtaposes the volatility of short-rate expectations at the two-year maturity with the volatility of term premia at the 10-year maturity (we present results for the maturities at which the contribution of the respective terms is the largest). The upper panel covers our entire sample period; the bottom panel zooms in on the events during and after the 2007/09 financial crisis. Compared to term-premium volatility, the volatility of short-rate expectations rises more abruptly when the economy enters into a recession and during distress in financial markets, as visible during the LTCM and the dot-com crises, and most clearly in the early stages of the 2007/09 financial crisis. These periods typically coincide with monetary easing by the Fed. Specifically, during easing episodes the volatility of short-rate expectations is about 50 basis points higher at the two-year maturity and 25 basis points higher at the 10-year maturity compared to other weeks in our sample.[21] The corresponding

[21] We define an easing (tightening) episode as the period during which the Fed lowered (increased) the Fed funds target in a sequence of at least three consecutive moves. Out of 999 weeks in our sample, we classify 202 weeks as tightenings and 270 weeks as easings.

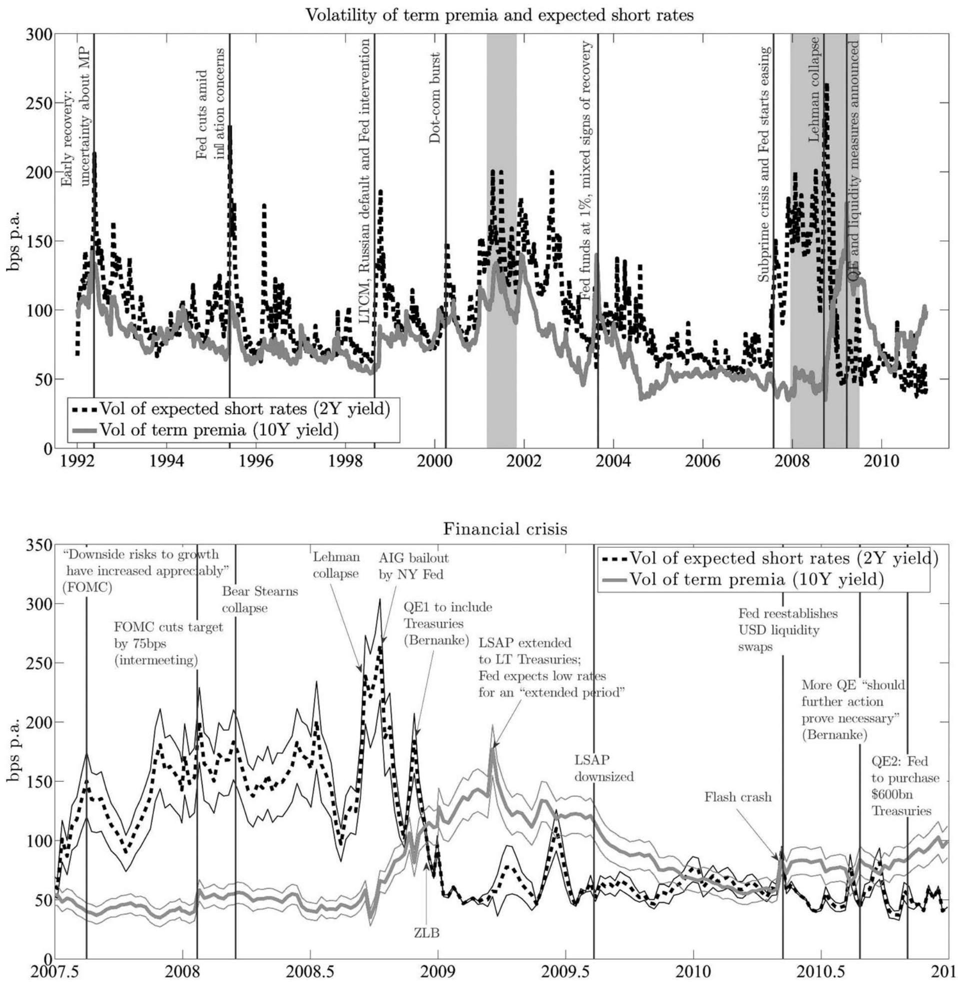


**Figure 5. Yield volatility and selected events.** The figure plots the model-implied volatility of short-rate expectations at the two-year maturity and the volatility of term premia at the 10-year maturity. Shaded areas are the NBER-dated recessions. Vertical lines mark selected economic or financial events. The upper plot spans the entire sample period; the bottom plot looks at the subperiod of the financial crisis from June 2007 through the end of our sample in December 2010. The thin lines in the bottom panel denote the 95% confidence interval.

increase in term premium volatility during easings is less than four basis points at the two-year maturity and nine basis points at the 10-year maturity.

### *A.3. Yield Volatility during the 2007/09 Financial Crisis*

The 2007/09 financial crisis illustrates the distinct behavior of interest rate volatility along the term premium and short-rate expectations dimensions.

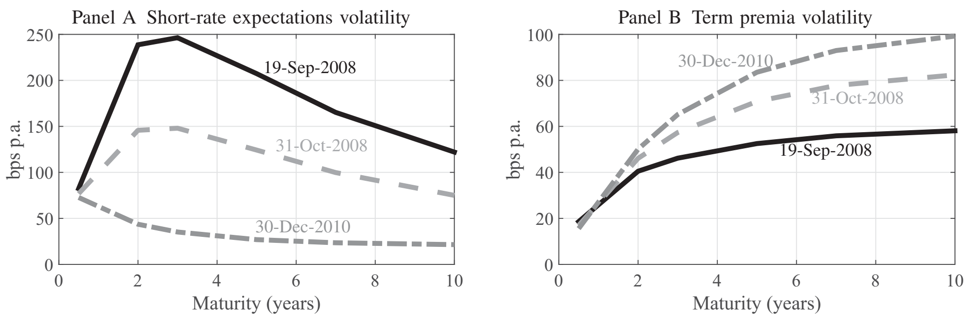


**Figure 6. Term structures of conditional yield volatilities.** Panel A presents the volatility of short-rate expectations across maturities on three selected dates. Panel B reports analogous results for the volatility of term premia.

The bottom panel of Figure 5 focuses on the subperiod between June 2007 and December 2010 and superimposes volatility dynamics with a number of events during that interval.

The volatility of short-rate expectations increases in the summer of 2007 when the first problems in the subprime market emerge, reaching a peak at the time of the Lehman collapse and AIG bailout. It is visibly reduced after the Fed implemented the zero lower bound (ZLB) policy in December 2008, communicating its intension to keep the federal funds rate low for "some time." The volatility of term premia shows a different pattern, remaining low until the fall of 2008. The AIG bailout marks the point when the premium volatility begins a persistent rise. The consistently high levels of premium volatility starting from late 2008 line up with the QE measures undertaken by the Fed (see, e.g., Fawley and Neely (2013) for the dating of those events). For instance, the Fed's announcement to extend its Large Scale Asset Purchases (LSAP) program to long-term Treasuries on March 18, 2009 coincides with the week in our sample in which the premium volatility reaches the highest level on record. It remains elevated though August 2009, when the FOMC announced a planned slowdown in LSAP. After this point, which was followed by additional announcements pointing to a slowdown in LSAP, the premium volatility begins a several-month-long decline until mid-2010, when the Fed announced a new set of liquidity measures.

To highlight how the shape of the entire term structure of conditional volatility can change over time, we study its dynamics on three dates: September 19, 2008, the week that Lehman Brothers filed for bankruptcy; October 31, 2008, the week after the Fed and the Treasury Department introduced measures to support liquidity in financial markets; and December 30, 2010, the end of our sample, at which point the second round of QE was already in place. Figure 6 presents the point estimates of conditional volatilities on each of these days as a function of maturity. The figure shows how a progressively lower volatility of short-rate expectations on these dates is accompanied by a successive rise in the volatility of term premia across maturities.

While most of the literature focuses on the effects of Fed policies on the level of yields, our results suggest that they have had a significant impact on yield volatilities as well. Our evidence is consistent with the view that the Fed has been successful in reducing the volatility of market expectations about the path of the short rate, and that the QE measures have worked mostly through their influence on the risk premia (e.g., Bernanke (2013)).

### *B. Effects of Volatility on the Conditional Distribution of Interest Rates*

The above results point to a nontrivial effect of stochastic volatility on the conditional distribution of interest rates. Using the same dates as in the previous section, Figure 7 presents the model-implied conditional distribution of the five-year yield one month ahead (Panel A), the corresponding term premium and expectations components (Panels C and D), and their respective volatilities (Panels E and F). For comparison, Panel B displays the yield distribution assuming that the conditional volatility is constant (we fix the conditional covariance matrix, $V_t$, at its sample average). Each plot is constructed by Monte Carlo simulations with 10,000 replications.

Panel A indicates that the conditional distribution of interest rates has undergone dramatic changes, not only in conditional means but also in the uncertainty surrounding them. The model implies that the distribution of the five-year yield moved from being widely spread around 3.1% in the week of the Lehman collapse to being tightly centered around 2.1% at the end of 2010. Exploring the sources of this shift, Panels C and D suggest that they stemmed mainly from changing short-rate expectations whose volatility shrank noticeably during the final two years of our sample. In contrast, over the same time span, the distribution of the term premium has widened but not sufficiently so to supersede the effect that the decline in short-rate expectations volatility had on the overall conditional distribution of yields. Panels E and F emphasize these dynamics by looking at the conditional distribution of volatilities. Specifically, Panel F makes clear that from the time of the Lehman collapse through the end of 2010 the volatility of short-rate expectations declined significantly, and at the end of our sample it was expected to stay at historically low levels. Comparing these results with Panel B of Figure 7, it is clear that, due to the constant volatility assumption, a Gaussian model would have missed these aspects of interest rate dynamics.

### *C. Comovement between Shocks to Term Premia and Short-Rate Expectations*

By modeling directly the stochastic covariance of risks in the yield curve, our setting is suited to analyze the question of how shocks to short-rate expectations and shocks to term premia in Treasury yields co-vary over time. Such correlation is interesting not only from an asset pricing but also from a monetary policy perspective. For example, a negative conditional correlation could suggest that the Fed faces a trade-off between easing monetary policy, for instance, to stimulate the economy, and increasing the inflation risk premium.

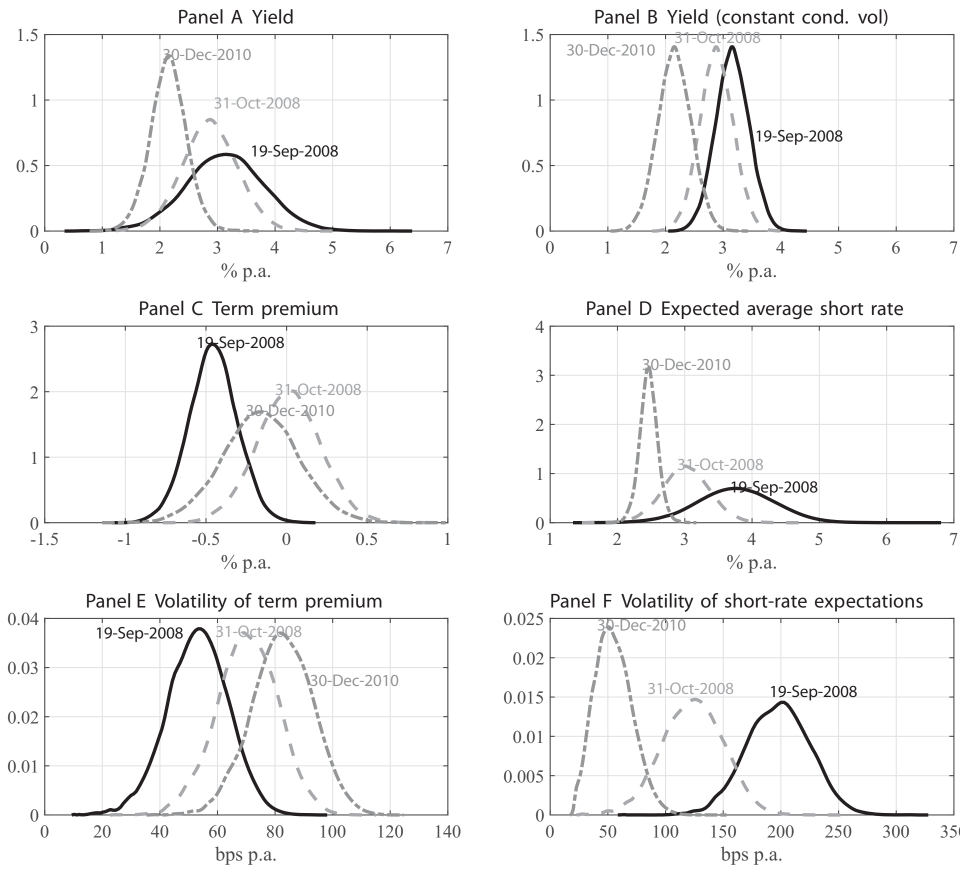


**Figure 7. Conditional distribution of the five-year yield and its volatility at a one-month-ahead horizon.** Panels A and C through F are generated from our baseline model, and describe the one-month-ahead distribution (kernel density) of the five-year yield and of its second moments at three dates in our sample. Panel A shows the conditional distribution of the five-year yield, Panels C and D show the conditional distribution of the term premium and the expected short rate, and Panels E and F show the distribution of the corresponding volatilities. Panel B displays the conditional distribution of the five-year yield assuming that the volatility is constant throughout our sample, fixing the $V_t$ matrix at its sample average. Each density is obtained from 10,000 Monte Carlo replications.

The instantaneous correlation between shocks to term premia and shocks to short-rate expectations is $v_t^{E,TP,\tau}/\sqrt{v_t^{E,\tau} \times v_t^{TP,\tau}}$. Figure 8 plots its time series for the five-year yield. The correlations at other maturities behave similarly and are summarized in the Internet Appendix. Two features are worth highlighting. First, the correlation is on average close to zero (we cannot reject the hypothesis that the conditional correlation is zero on average). Second, the correlation is

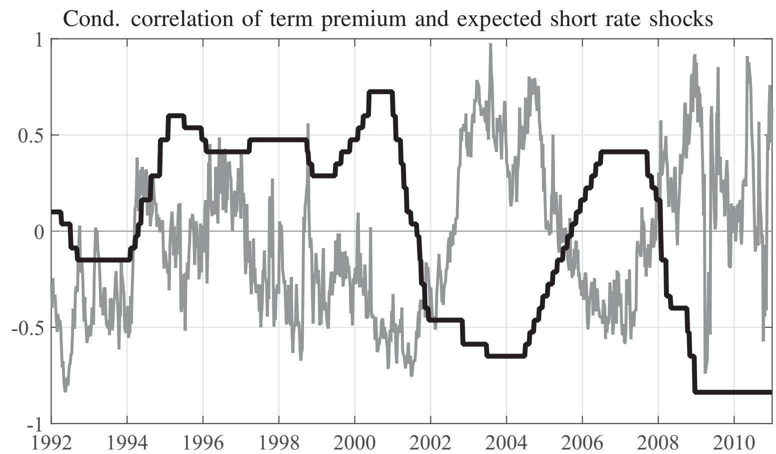


**Figure 8. Correlation between shocks to short-rate expectations and to term premia.** The figure describes the instantaneous correlation between shocks to short-rate expectations and to term premia for the five-year yield. The solid stepped line superimposes the (rescaled) Federal funds rate target.

persistent and clearly time-varying; it changes signs multiple times during our sample and has a sample standard deviation of about 0.4.[22]

The correlation between shocks to term premia and expected short rates increases during easings and declines during tightenings. Figure 8 superimposes the correlation with the evolution of the Fed funds target (the unconditional correlation between the two series in the plot is $-0.5$). The model suggests that Greenspan's conundrum period, that is, the lack of a response of long-term yields to the Fed's 2004/06 tightening, was associated with a gradual decline in the correlation between expected short-rate shocks and term premium shocks from 0.8 to $-0.4$. More generally, the low average correlation between those shocks is related to the observation that in the last two decades the link between short- and long-term interest rates has been largely severed (e.g., Thornton (2012)). It also casts light on the conclusion in recent literature that the Fed is able to influence the expectations of the short rate at long horizons, as suggested by the strong impact of monetary policy shocks on long-term yields (Nakamura and Steinsson (2013), Hanson and Stein (2015)). Our results

[22] To verify these results with a proxy independent of our model, we use an approach motivated by Hanson and Stein (2014). They argue that changes in the two-year yield are dominated by short-rate news, whereas changes in long-term yields that are (unconditionally) orthogonal to changes in the two-year yield should mostly reveal term premia news. Accordingly, we regress $\Delta_{1d} y_t^{(10)} = a + b\Delta_{1d} y_t^{(2)} + \varepsilon_t^{(10)\perp}$ on the full sample, where $\Delta_{1d}$ denotes a daily change, and compute the rolling 60-day correlation between $\Delta_{1d} y_t^{(2)}$ and $\varepsilon_t^{(10)\perp}$. While by construction the unconditional correlation is zero, the rolling correlation varies significantly over time: it has a standard deviation of 0.35, and its sample 25[th] and 75[th] percentiles are $-0.25$ and 0.26, respectively, which is broadly consistent with our model-based results.

indicate that while unconditionally the correlation between term premia and expected short rates is low, it can increase dramatically when the Fed is easing.

## V. Implications of Yield Volatility for Risk Compensation in the Treasury Market

The preceding results raise the question of how volatility risk is priced in Treasury bonds and in fixed income market more broadly.

### A. Sharpe Ratios for the Yield and Variance Shocks

Using expression (12), bond risk premia have two components loading on the yield curve states $Y_t$ and the volatility states $V_t$,

$$brp_t^\tau = brp_t^\tau|\mathrm{d}W_t^{\mathbb{P}} + brp_t^\tau|\mathrm{d}Z_t^{\mathbb{P}}, \tag{24}$$

where the first term on the right-hand side is the compensation for facing the yield curve shocks ($\mathrm{d}Z_t^{\mathbb{P}}$) and the second term is the compensation for facing the volatility shocks ($\mathrm{d}W_t^{\mathbb{P}}$). Similarly, we decompose the instantaneous variance of yields in equation (13) into variance induced by $\mathrm{d}Z_t^{\mathbb{P}}$ and $\mathrm{d}W_t^{\mathbb{P}}$ shocks:

$$v_t^\tau = v_t^\tau|\mathrm{d}W_t^{\mathbb{P}} + v_t^\tau|\mathrm{d}Z_t^{\mathbb{P}}. \tag{25}$$

The variance of the instantaneous bond returns is $\tau^2 v_t^\tau$. The associated bond market Sharpe ratio, that is, premium earned per unit of risk, for facing both types of shocks is

$$SR_t^\tau = \frac{brp_t^\tau}{\tau\sqrt{v_t^\tau}}. \tag{26}$$

We can define Sharpe ratios associated with a specific source of risk as

$$SR_t^{\tau,Z} = \frac{brp_t^\tau|\mathrm{d}W_t^{\mathbb{P}}}{\tau\sqrt{v_t^\tau|\mathrm{d}W_t^{\mathbb{P}}}} \tag{27}$$

$$SR_t^{\tau,W} = \frac{brp_t^\tau|\mathrm{d}Z_t^{\mathbb{P}}}{\tau\sqrt{v_t^\tau|\mathrm{d}Z_t^{\mathbb{P}}}}, \tag{28}$$

where $SR_t^{\tau,Z}$ is the compensation per unit of yield curve risk induced by $\mathrm{d}Z_t^{\mathbb{P}}$ shocks, keeping shocks to yield volatility $\mathrm{d}W_t^{\mathbb{P}}$ fixed, and vice-versa for $SR_t^{\tau,W}$. To show the effect of the time-varying second moments, we also construct

$$\overline{SR_t^\tau} = \frac{brp_t^\tau}{\tau\sqrt{\overline{v^\tau}}}, \tag{29}$$

which reflects the variation in the bond risk premium, keeping the instantaneous variance constant at its sample average $\overline{v^\tau}$.

The sample average of the total bond market Sharpe ratio, $SR_t^\tau$, is about 0.1 across maturities.[23] We find that both $\overline{SR_t^\tau}$ and $SR_t^{\tau,Z}$ are very highly correlated with the total bond market Sharpe ratio, $SR_t^\tau$. The sample correlation between $\overline{SR_t^\tau}$ and $SR_t^\tau$ is above 0.95 across maturities, and the sample correlation between $SR_t^{\tau,Z}$ and $SR_t^\tau$ is effectively one. These results suggest that the variation in Sharpe ratios for Treasury bonds is dominated by the time-varying market prices of risk and that investors holding bonds require compensation almost exclusively for the yield curve shocks. Such interpretation is consistent with evidence that yields are largely insensitive to volatility shocks (Collin-Dufresne and Goldstein (2002)). The model implies that a one-standard-deviation shock to volatility states has an effect of less than five basis points on the 10-year interest rate, and an even smaller effect at shorter maturities.[24] Thus, since yields are largely insensitive to volatility shocks, those shocks have little effect on the expected excess bond returns.

These findings are important for understanding how our model circumvents Duffee's (2010) critique that high-dimensional settings (with four or five Gaussian factors) imply implausibly high conditional Sharpe ratios as a consequence of in-sample overfitting. Although our model has six sources of risk, in terms of pricing shocks to Treasury bonds, it behaves like a low-dimensional Gaussian model. The additional flexibility provided by the volatility states is used to fit the second-moment information that we supply in estimation.

The implication that volatility is not directly related to Treasury risk premia agrees with the evidence that the link between various measures of bond volatility and expected bond returns is weak (e.g., Le and Singleton (2013)). However, our model does not imply that the compensation for variance risk is zero in all fixed income assets. This is revealed by the properties of $SR_t^{\tau,W}$, which measures the compensation for volatility risk per unit of this specific risk, rather than per unit of the overall yield curve risk. Figure 9 depicts the time-series dynamics of $SR_t^{\tau,Z}$ and $SR_t^{\tau,W}$ for the two- and 10-year bonds.

[23] Unconditional annual Sharpe ratios computed using CRSP bond returns over a long sample (1958 to 2010) are between 0.44 (for maturities between 12 and 24 months) and 0.28 (for maturities between 60 and 120 months). During our sample period (1992 to 2010), bond risk premia (and term premia) have been hovering close to zero since mid-2000, as implied by our model and similar estimates in the literature. This can explain the low average model-implied Sharpe ratios. Our results parallel those in Kim and Singleton (2012), who report conditional Sharpe ratios below 0.1 for Japanese bonds in the low interest rate environment.

[24] Using equation (11), the contribution of the $Tr(C(\tau)V_t)$ term to model-implied yields is very small relative to the contribution of the $B(\tau)'Y_t$ term. At our estimates, the effect of $V_t$ on the cross-section of yields is comparable to the size of the measurement errors in yields, whose values are about seven to nine basis points (e.g., Bekaert, Hodrick, and Marshall (1997)). Thus, volatility states are effectively unspanned by the yield curve. Duffee (2011) argues that measurement error can break the exact mapping between yields and the state vector even when the theoretical model assumes spanning. Therefore, although in our model the volatility states are theoretically spanned by the cross-section of yields, in practice they cannot be recovered from yields given the size of the measurement error.

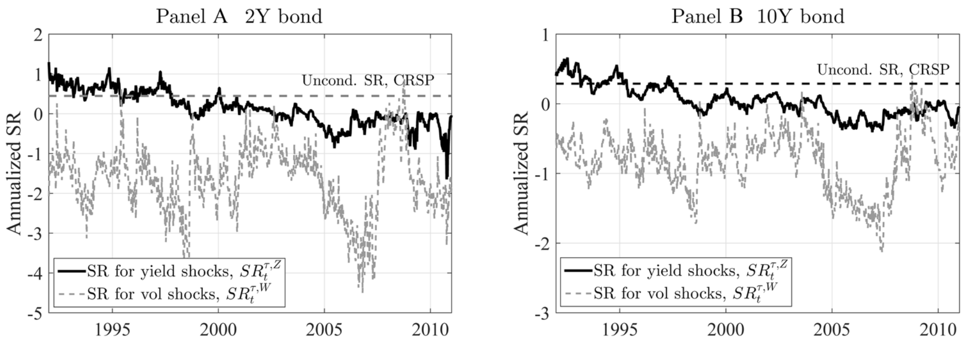


**Figure 9. Instantaneous Sharpe ratios.** The figure plots the instantaneous annualized Sharpe ratios from our baseline model for the two- and 10-year bonds. The Sharpe ratios are decomposed into the compensation for facing yield curve shocks and the compensation for facing variance shocks, according to expressions (27) and (28). The dashed lines mark the level of the unconditional annual Sharpe ratios for CRSP bond portfolios with maturities between 12 and 24 months (Panel A) and maturities between 60 and 120 months (Panel B), and are computed from a long historical sample (1958 to 2010) over which CRSP data are available.

The time-series behavior of $SR_t^{\tau,W}$ shows that the compensation for variance risk can be large and is strongly time-varying. The sign on $SR_t^{\tau,W}$ is generally negative, implying that investors are willing to pay a premium for protection against volatility risk. The positive convexity makes Treasury bonds, at least in theory, potential hedges against such risk. In practice, given that the model-implied effect of volatility risk on the yield curve is smaller than plausible measurement error, our model identifies the parameters of the variance risk premium not from yields but from the realized and implied variances provided in estimation. The average conditional (annualized) Sharpe ratios for volatility risk range between $-1.7$ at the two-year maturity to $-0.8$ at the 10-year maturity. These numbers suggest that it is costly to buy protection against variance risk, a result that is similar to existing evidence for the equity market. For instance, Egloff, Leippold, and Wu (2010) show that buying variance protection in the equity market has historically earned an annualized Sharpe ratio of about $-1.4$. Similar results are reported by Dew-Becker et al. (2015), who document a Sharpe ratio of $-1.7$ associated with hedging purely transitory shocks to the realized variance in the equity market.

### B. Compensation for the Variance Risk

The model allows us to ask more specific questions about the type of volatility shocks that investors are willing to pay such a high premium to hedge. Using the estimated dynamics of the volatility states $V_t$ under the physical and the risk-neutral measures, we can analyze whether it is shocks to the variance of

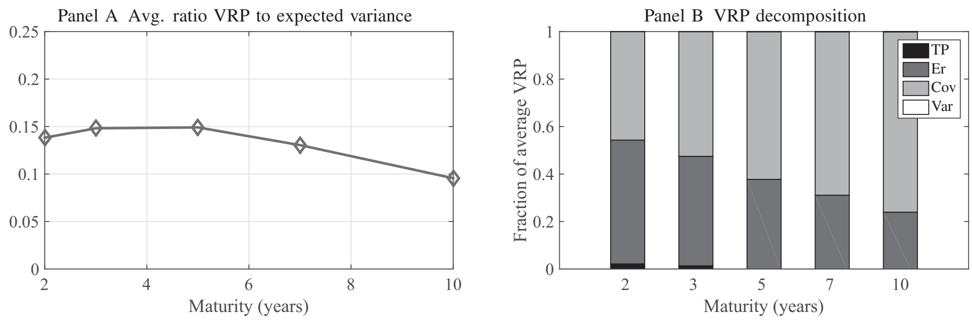


**Figure 10. Decomposition of the variance risk premium at different maturities.** Panel A shows the sample average of the ratio between the one-month-ahead variance risk premium and the expected variance at the same horizon, that is, $vrp^{\tau}_{t,t+1M}/v^{\mathbb{P},\tau}_{t,t+1M}$, for different yield maturities $\tau$. Panel B decomposes the average variance risk premium according to equation (30) into components: (1) average compensation for the variance of term premia ("TP"), (2) average compensation for the variance of short-rate expectations ("Er"), (3) average compensation for the covariance between term premia and short-rate expectations ("Cov"), and (4) average compensation for the vol-of-vol ("Var"). These components are expressed as fractions of the average variance risk premium. Thus, each bar in Panel B sums to one. The contribution of the "Var" component is close to zero and essentially invisible in the plot.

short-rate expectations, to the variance of term premia, or to the covariance between expectations and premia that induce such compensation.

Figure 10 summarizes the properties of the variance risk premium, defined in equation (15), as a function of the maturity of the underlying bond. For ease of interpretation, we follow the convention of presenting the average ratio of the variance risk premium to the expected variance (e.g., Carr and Wu (2009)). In Panel A, we report the sample average of $\frac{vrp^{\tau}_{t,t+h}}{E^{P}_{t}(v^{\tau}_{t,t+h})}$, with $h$ equal to one month ($h = 1/12$). This ratio ranges between 10% and 15% across maturities, is slightly humped at maturities of two to three years, and declines at longer maturities.

Using equation (23), one can obtain the following decomposition:

$$vrp^{\tau}_{t,t+h} = vrp^{E,\tau}_{t,t+h} + vrp^{TP,\tau}_{t,t+h} + 2vrp^{E,TP,\tau}_{t,t+h} + vrp^{C,\tau}_{t,t+h}, \tag{30}$$

where

$$vrp^{E,\tau}_{t,t+h} = \frac{1}{h}\int_0^h \left(E^{\mathbb{Q}}_t\left(v^{E,\tau}_{t,t+s}\right) - E^{\mathbb{P}}_t\left(v^{E,\tau}_{t,t+s}\right)\right) \mathrm{d}s. \tag{31}$$

The other terms in equation (30), all of which are an affine function of $V_t$, are constructed analogously. The term $vrp^{E,\tau}_{t,t+h}$ is the compensation that investors require for facing shocks to the variance of short-rate expectations, $vrp^{TP,\tau}_{t,t+h}$ is the compensation for facing shocks to the variance of the term premium, $vrp^{E,TP,\tau}_{t,t+h}$ is the compensation for facing shocks to the covariance between expectations and premia, and $vrp^{C,\tau}_{t,t+h}$ is the compensation for the "vol-of-vol" risk.

To understand the contribution of these risks, in Figure 10 we plot the fraction that each adds to the total variance risk premium. The fractions are computed as the sample average of each element on the right-hand side of (30) to the sample average $vrp^{\tau}_{t,t+h}$, $h = 1/12$. The numbers in the figure sum to one at any given maturity. The two largest sources, which jointly account for almost the entire variance premium, are $vrp^{E,\tau}_{t,t+h}$ and $vrp^{E,TP,\tau}_{t,t+h}$. The contribution of $vrp^{E,\tau}_{t,t+h}$ declines with maturity, while that of $vrp^{E,TP,\tau}_{t,t+h}$ increases with maturity. The model suggests that the compensation for facing shocks to the variance of the term premium, $vrp^{TP,\tau}_{t,t+h}$, per se is small, and that the premium for the vol-of-vol component, $vrp^{C,\tau}_{t,t+h}$, is effectively zero. Thus, investors pay a premium for being able to hedge the risks that short-rate expectations become more volatile and that shocks to short-rate expectations propagate onto the long end of the yield curve. The latter effect is intuitively captured by shocks to the covariance between short-rate expectations and premia. To the extent that the volatility of short-rate expectations can be linked to uncertainty about the future path of monetary policy, these results suggests that there is significant demand for protection against this source of uncertainty.

## VI. Robustness and Extensions

### A. Comment on the ZLB

One caveat concerning the above results is that our model does not impose the ZLB constraint on the short-term interest rate, despite the Federal funds rate being effectively at the ZLB for the last two years of our sample. Recent models that incorporate the ZLB are predominantly Gaussian.[25] By construction, such models generate an asymmetric distribution of the short rate at the ZLB, but do not account for the tightening of the yield distribution that we document in Section IV.B. Ideally, one would combine our model with a ZLB restriction. However, recognizing the challenges associated with an empirical implementation of such an extension, there are reasons to expect that our volatility decomposition is not significantly affected by the ZLB. First, although the ZLB is clearly important for the very short end of the yield curve, interest rates with maturities of one or more years have remained substantially above zero though the end of our sample. Swanson and Williams (2014) find that Treasury yields with one or more years to maturity are as responsive to news over the 2008 to 2010 period as in the earlier part of the sample. Similarly, Bauer and Rudebusch (2014) argue that the tightness of the ZLB constraint increased significantly only after August 2011, when the Fed provided a dated statement promising to keep interest rates near zero for the next two years. Prior to that announcement, market participants expected the liftoff from the

[25] See Bauer and Rudebusch (2014) for a summary of the recent literature. One exception is Kim and Singleton (2011), who estimate ZLB models with stochastic volatility on Japanese yields. Since their focus is on two-factor models, they conclude that more factors are needed to fit yields and volatilities jointly.

ZLB to occur within several months. Second, and related, our use of survey forecasts requires that the model-implied expectations of the short-term interest rate be consistent with those of professional forecasters. This mitigates the concern that the model may interpret the entire variation in yields in the recent episode as stemming from the term premium. Our estimates show that, even at the two-year maturity, short-rate expectations remained relatively volatile, with conditional volatility around 50 basis points at the end of 2010. These arguments suggest that the volatility decomposition at maturities of two or more years, which are our main focus, should not be strongly affected by the ZLB in the period of our study. We leave an extension of our model to incorporate the ZLB to future work.

### *B. Relative Contribution of the Model versus Second-Moment Data*

Our analysis departs from the existing literature on two dimensions: the model for conditional second moments of yields and the use of more informative second-moment data. It is therefore important to disentangle the relative contributions of these two dimensions to our results. To assess the contribution of the data, we estimate our baseline model on yields and interest rate surveys without providing information about second moments. Details on this specification are in the Internet Appendix, but here we summarize the main results. The model-implied conditional variances in this yields-only setting vary little over time and in a way that is unrelated to the observed volatility dynamics. That is, there is not enough information in yields themselves to identify the conditional volatility dynamics. This conclusion is consistent with earlier studies that rely only on yields to estimate term structure models with stochastic volatility, finding either little or counterfactual variation in the model-implied conditional second moments (Collin-Durfesne, Goldstein, and Jones (2009), Jacobs and Karoui (2009)). As such, our use of informative second-moment data is critical for providing a reliable decomposition of yield volatility.

### *C. Comparison with the $A_2(4)$ Model*

It is worth discussing how our approach to modeling second moments in yields relates to the standard affine framework. Given that ATSMs with a single volatility state are known to face a tension in matching volatilities at the short and the long ends of the yield curve, we estimate an $A_2(4)$ model. The specification we adopt follows Joslin (2010) except that, in analogy to equation (7), we assume that the short rate is a function of only conditionally Gaussian factors. We estimate the model using the same techniques, yields, and second-moment measurements as in our baseline case.[26] The model has four

[26] In practice, we find that not allowing the volatility factors to enter the short rate prevents the estimation from landing in a local minimum. Estimating a version of the model in which volatility affects the short rate, we find that, while it generates lower RMSEs for yields, the volatilities it implies are either uncorrelated or negatively correlated with the second-moment data used in

factors in total, $F_t = (F_t^{(1)}, F_t^{(2)}, F_t^{(3)}, F_t^{(4)})'$, of which the first two factors are square-root processes and the last two are conditionally Gaussian. The instantaneous covariance matrix of $F_t$ is specified as

$$\frac{1}{\mathrm{d}t}\langle \mathrm{d}F_t \rangle = H_0 + H_1^{(1)} F_t^{(1)} + H_1^{(2)} F_t^{(2)}, \tag{32}$$

where $H_0$, $H_1^{(1)}$, and $H_1^{(2)}$ are $4 \times 4$ matrices defined such that they allow for instantaneous correlations between shocks to conditionally Gaussian factors while maintaining independence of shocks to the square-root factors. The maximally flexible specification of (32) involves an estimation of nine parameters in matrices $H_0$, $H_1^{(1)}$, and $H_1^{(2)}$.

Table V, Panel A summarizes the cross-sectional and time-series fit of the $A_2(4)$ model to the conditional second moments of yields. The layout of the results is analogous to Table II. The $A_2(4)$ model generates lower RMSEs for the realized variances and covariances (by about six basis points on average) compared to our baseline specification. At the same time, it also fails more often in the cross-sectional specification test in regression (16).

For most volatility measurements, the null hypothesis of zero intercept and slope coefficients is rejected at the 5% level. To understand the difference between the two models, we note that, relative to our volatility specification in (3), the $A_2(4)$ model has additional parameters that govern the covariance matrix of the yield factors in (32). This allows us to closely match the realized second moments, including the jagged high-frequency component of their dynamics, reducing the RMSEs. However, the results of regression (16) suggest that this comes at the cost of missing some of the lower-frequency movements in yield volatility. This property becomes visible in the dynamic fit summarized in Panel B of Table V. Estimating regression (17) for volatility forecast errors in the $A_2(4)$ model, we find that the null hypothesis of all coefficients being zero is strongly rejected at the one-week horizon, and is rejected at the 5% level in several instances at longer horizons. Different from our baseline model, which features relatively stable $\bar{R}^2$s across forecast horizons (Table II, Panel B), adjusted $R^2$s in regressions based on the $A_2(4)$ model are high for the one-week horizon and decline at longer horizons. This is because, as the forecast horizon increases, the $A_2(4)$ model generates forecast errors that are skewed to the right—a consequence of a fast mean-reversion in the volatility dynamics. Indeed, Figure 11, which plots the conditional variance forecasts for the 10-year yield from the baseline model and the $A_2(4)$ model, shows that longer horizon forecasts are less volatile in the latter case.

estimation, thus echoing results in earlier literature. Intuitively, the unrestricted estimation uses the flexibility provided by volatility factors to fit the cross-section of yields. We also find that including survey expectations as measurements decreases the fit to volatilities. This suggests that more than two conditionally Gaussian factors are necessary to describe the first moments of yields. Here, we focus on the estimation of the $A_2(4)$ model that gives the best fit to the conditional second moments of yields, and leave its extensions to future research. Additional details can be found in the Internet Appendix.

**Table V**

## Volatility Fit of the $A_2(4)$

Panel A summarizes the cross-sectional fit to realized variances, realized covariances, and implied variances by the $A_2(4)$ model. Rows (1) and (2) report the RMSEs for realized variances ($\times 10^4$) and realized volatilities (in basis points). For the few data points where covariances become negative (seven observations in total across all measurements), to compute the RMSE in row (2) we take the square root of the absolute value of the measurement. Rows (3) though (5) are based on regression (16). Row (3) reports $p$-values testing the null hypothesis that the regression intercept is zero $\alpha_0 = 0$; row (4) shows $p$-values for the Wald test of the null that all regression coefficients are zero $(\alpha_0, \alpha_1')' = 0$. * indicates rejection of the null at the 5% level. The Wald test does not use any adjustment of the parameter covariance matrix. Row (5) provides adjusted $\bar{R}^2$s. Panel B reports regressions of the model-implied forecast errors specified in equation (17). The forecast errors are computed for horizons $h$ of one, four, and eight weeks ahead. Row (1) reports $p$-values testing the null that the regression intercept is zero $\beta_0 = 0$; row (2) shows $p$-values for the Wald test of the null that all regression coefficients are zero $(\beta_0, \beta_1')' = 0$. * indicate rejection of the null at the 5% level. The Wald test is based on the Newey–West adjustment with $h - 1$ lags. Row (3) reports adjusted $\bar{R}^2$s.

Panel A. Cross-Sectional Fit

| | Realized Variance | | | | | Realized Covariance | | | Implied Variance | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2Y | 3Y | 5Y | 7Y | 10Y | 2,5Y | 2,10Y | 5,10Y | 5Y | 10Y |
| (1) RMSE Var $\times 10^4$ | 0.63 | 0.81 | 0.54 | 0.34 | 0.49 | 0.46 | 0.32 | 0.29 | 0.25 | 0.19 |
| (2) RMSE Vol (bps) | 20.72 | 22.45 | 14.22 | 11.53 | 15.98 | 16.38 | 13.31 | 12.67 | 9.29 | 8.17 |
| (3) pval $(\alpha_0 = 0)$ | 0.00* | 0.00* | 0.00* | 0.01* | 0.01* | 0.22 | 0.75 | 0.00* | 0.86 | 0.12 |
| (4) $p$-value $(\alpha_0, \alpha_1')' = 0$ | 0.00* | 0.00* | 0.00* | 0.45 | 0.39 | 0.00* | 0.00* | 0.02* | 0.00* | 0.00* |
| (5) $\bar{R}^2$ | 0.23 | 0.14 | 0.05 | 0.13 | 0.11 | 0.15 | 0.18 | 0.09 | 0.26 | 0.20 |

Panel B. Dynamic Fit

| | Realized Variance | | | | | Realized Covariance | | |
|---|---|---|---|---|---|---|---|---|
| | 2Y | 3Y | 5Y | 7Y | 10Y | 2,5Y | 2,10Y | 5,10Y |
| $h = 1$ week | | | | | | | | |
| (1) $p$-value $\beta_0 = 0$ | 0.01* | 0.00* | 0.02* | 0.04* | 0.03* | 0.11 | 0.13 | 0.65 |
| (2) $p$-value $(\beta_0, \beta_1')' = 0$ | 0.00* | 0.00* | 0.00* | 0.00* | 0.00* | 0.00* | 0.00* | 0.00* |
| (3) $\bar{R}^2$ | 0.17 | 0.13 | 0.13 | 0.16 | 0.12 | 0.16 | 0.24 | 0.18 |
| $h = 4$ weeks | | | | | | | | |
| (1) $p$-value $\beta_0 = 0$ | 0.06 | 0.01* | 0.08 | 0.15 | 0.12 | 0.19 | 0.10 | 0.90 |
| (2) $p$-value $(\beta_0, \beta_1')' = 0$ | 0.00* | 0.00* | 0.06 | 0.57 | 0.71 | 0.00* | 0.00* | 0.33 |
| (3) $\bar{R}^2$ | 0.05 | 0.04 | 0.03 | 0.04 | 0.04 | 0.03 | 0.02 | 0.03 |
| $h = 8$ weeks | | | | | | | | |
| (1) $p$-value $\beta_0 = 0$ | 0.22 | 0.12 | 0.48 | 0.77 | 0.70 | 0.45 | 0.28 | 0.66 |
| (2) $p$-value $(\beta_0, \beta_1')' = 0$ | 0.04* | 0.10 | 0.36 | 0.82 | 0.90 | 0.13 | 0.26 | 0.79 |
| (3) $\bar{R}^2$ | 0.03 | 0.02 | 0.02 | 0.04 | 0.03 | 0.02 | 0.01 | 0.04 |

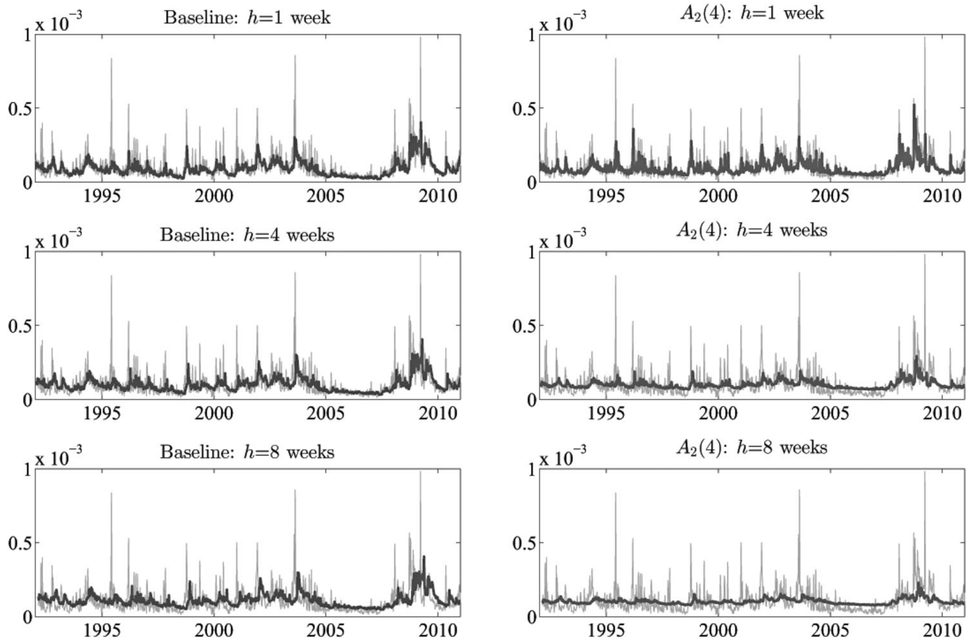


**Figure 11. Dynamic fit to the realized variance at different forecast horizons.** The figure presents the model-implied forecasts (solid line) of the realized variance of the 10-year yield at horizons $h$ of one, four, and eight weeks ahead. The thin line represents the data. The distance between the two lines measures the size of the forecast error. The left-hand panels present results for our baseline model; the right-hand panels present results for the $A_2(4)$ model.

Another difference between our model and the $A_2(4)$ model is in how they introduce correlations between state variables, and thus between objects within the yield curve, such as term premia and short-rate expectations. In Section IV.C, we argue that the correlation between premia and expectations is quite volatile and can frequently switch signs. In the $A_2(4)$ model, for the correlation between shocks to factors and other model-derived objects to change signs, the loadings on $F_t^{(1)}$ and $F_t^{(2)}$ need to have opposite signs. While the $A_2(4)$ model generates a conditional correlation between term premium and short-rate expectations shocks that is low on average ($-0.12$ across maturities), it is significantly less variable than that implied by our baseline estimates. With a sample standard deviation of 0.05, the correlation stays relatively flat over time and does not display a business cycle pattern.

## VII. Conclusions

We decompose the conditional interest rate volatility in the U.S. Treasury market into volatilities of short-rate expectations, term premia, and their conditional covariance. To this end, we propose a no-arbitrage framework with stochastically correlated risks that we estimate with extensive data describing the dynamics of the second moments of yields. Short-rate expectations are the main source of volatility at the short end of the term structure, and volatile term premia dominate at the long end. Shocks to term premia and short-rate expectations comove in a nontrivial way. Their correlation is close to zero on average, but it is also quite volatile and varies negatively with the stance of monetary policy.

We study the behavior of interest rate volatility over the business cycle and during the recent financial crisis. Over the last two decades, short-rate expectations become increasingly volatile as the economy enters into a recession, during times of distress in asset markets, and when the Fed eases monetary policy. While term premium volatility also tends to increase in those episodes, its reaction is relatively more muted. A case that illustrates the distinct properties of the term premium and short-rate expectations volatilities is the 2007/09 financial crisis and its aftermath. Our decomposition supports the view that, during that period, the Fed was successful in reducing the volatility of expectations about the future path of the short rate, and that the unconventional QE measures have led to an increase in term-premium volatility.

Our setting allows us to study the compensation for facing yield curve shocks and volatility shocks in fixed income markets. We find that investors in Treasury bonds are compensated for taking on yield curve risk but are not exposed to shocks to interest rate volatility. At the same time, the model implies that investors are willing to pay a large premium for hedging volatility risk though interest rate derivatives. The main source of the variance risk premium is the exposure to the variance that arises from fluctuations in short-rate expectations rather than term premia.

Our analysis can be extended in several directions. In obtaining the above results we have relied on a reduced-form model. This approach allows us to study economically well-defined yet endogenous objects—short-rate expectations and term premia, and their corresponding volatilities. A natural next step would be to examine the exogenous sources of interest rate volatility, its relation to macroeconomic fundamentals, and its real and nominal determinants. Our findings can also be used to motivate further study of how financial markets price monetary policy uncertainty. For instance, to the extent that the volatility of short-rate expectations can be linked to the Fed's policy, our results suggest a nontrivial premium that is earned for selling protection against monetary policy uncertainty. We leave investigation of these and related questions to future research.

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## Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

**Appendix S1:** Internet Appendix.